For the use of exterior form in daily physics, an introduction without coordinate frame.

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This is a short introduction of the exterior form formalism focus on its applications in physics and then mostly aimed to physics students. If exterior forms are more than a century old they are unfortunately still seen (and teached) as a high level mathematics object and then little used outside theorical physics. We then focus here on simple examples which occure in daily phiysics. There exists already a lot of very good mathematical textbooks and courses on the subject but the originality of these notes, the physical applications aside, is that we keep a completely geometrical approach. As a rule of a game played here we never use a coordinate frame neither in the definitions nor in the proofs but only at the end in order to recover the classical physics equations. This approach is unusual but we think is helpful for the understanding and very valuable to grab the physical meanings of the mathematical object. To say differently we will always prefer the following

$$\text{Geometric definition} \; > \; \text{Coordinate definition}$$

$$\text{Geometric proof} \; > \; \text{Computational proof.}$$

A large part of these notes are just "notations rewriting" of well known physical objects but this should not be underestimate as it gives short and elegant expressions that are useful both for insights and computations. Appart from the *game* explained above most the material presented here is very well known [Fra11, Fla63, BBB85] with exeption maybe of Corollary 24 for which we have surprisingly not found its statement in the litterature and the discussion around Corollaries 19, 20 which we believe deserves more publicity. Hopefully the approach presented here could be generalized in a very abstract setting.

1 Exterior forms

1.1 Submanifolds as elementary objects

We denote Ω as the "physical universe" and we will always think Ω as a 3 or 4 dimensional manifold ($\mathbb{R}^3$ or $\mathbb{R}^4$ for example). Our basic object will be submanifold of Ω

$$\mathcal{V} \subset \Omega$$

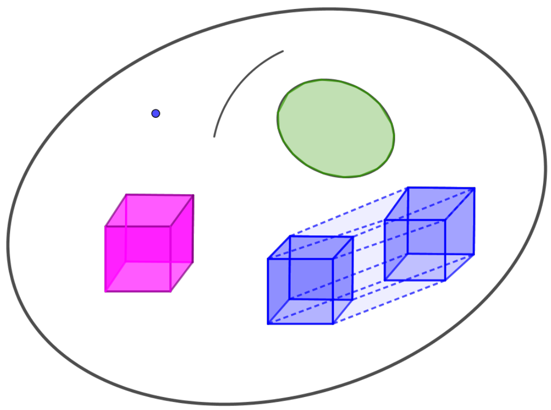

Figure 1: What can we do if we just have submanifolds and no coordinate frame ?

In any textbook the next step is to attach to Ω a set of local coordinate. This is the step we will try to avoid here, or at least we will try to write down as much as we can without any choice of local coordinate. For example if one consider the church "Notre Dame", one could say that it is at 48° North-2° East on the surface of the Earth but one could also say that the church is just a volume embedded in the universe $\mathcal{V}_{\text{Notre Dame}} \subset \Omega$ and this is a more fundamental mathematical representation. In these notes we ask the following.

Question. *If we restrict ourself to work only with submanifold without local coordinate, what could we do from a mathematical point of view?*

Here we will use exterior forms but will try to avoid Grassmann algebra as well.

1.2 Integration on a submanifold

Usually, introduction about exterior form start at the coordinate level giving the formal definition of an exterior algebra [Pau07, God70, LCC+09]. However, it is not so obvious at first sight why such an object is usefull and interesting in physics. So we would instead propose the following definition to go immediatly to the main point [Fla63].

Definition 1. A $k-$form is *what* to be integrated on a $k-$dimension submanifold.

This definition is made vague and unformal on purpose. Because it much closer of what any physics would want as a mathematical object. That is to choose an submanifold[1] : a path, a surface, a volume or a time interval $\times$ a volume and have physical quantity to integrate on it. This should really be though as the "meaningful physical quantity", as it doesn't depend on the choice of coordinate, not even the space-time metric g. If one think of a glass of water the quantity of water in the volume defined by the glass (a 3 dimension submanifold) should exist and be independant on the methods we measure space or time. From the lowest mathematical level this is just an application α that for any $k-$dimension submanifold $\mathcal{V}$ associate a real number $\alpha(\mathcal{V}) \in \mathbb{R}$.

$$\begin{array}{rcl} \alpha : k\text{-dimension submanifold} & \rightarrow & \mathbb{R} \\ \mathcal{V} & \rightarrow & \alpha(\mathcal{V}) \end{array}$$

We also denote

$$\alpha(\mathcal{V}) = \int_{\mathcal{V}} \alpha$$

but this notation is best to be seen as the definiton of the r-h-s than of the l-h-s.

Here are some examples in the case Ω is 3 dimensional.

dimension	basis	quantity	unit
0-form	1	Temperature	K
		Pressure	Pa
		Potential	V
		Reflective index	1
1-form	dx, dy, dz	Temperature gradient	$K.\boldsymbol{m}^{-1}$
		Pressure gradient	$Pa.\boldsymbol{m}^{-1}$
		Electric field	$V.\boldsymbol{m}^{-1}$
		Vector potential	$Vs.\boldsymbol{m}^{-1}$
		Magnetizing field	$Cs^{-1}.\boldsymbol{m}^{-1}$
2-form	$dx \wedge dy$, $dx \wedge dz$, $dy \wedge dz$	mass flow	$kg.s^{-1}.\boldsymbol{m}^{-2}$
		charge flow	$C.s^{-1}.\boldsymbol{m}^{-2}$
		radiation	$W.\boldsymbol{m}^{-2}$
		magnetic field	$Vs.\boldsymbol{m}^{-2}$
		Electric displacement field	$C.\boldsymbol{m}^{-2}$
3-form	$dx \wedge dy \wedge dz$	density of mass	$kg.\boldsymbol{m}^{-3}$
		density of charge	$C.\boldsymbol{m}^{-3}$
		density of energy	$J.\boldsymbol{m}^{-3}$
		heat capacity	$JK^{-1}.\boldsymbol{m}^{-3}$

In $\mathbb{R}^3$ the dimension of the $k-$form $\Lambda^k(\mathbb{R}^3)$ are respectively $1, 3, 3$ and 1. The main message here is that exterior form are natural to describe daily physics quantities and are not more complicated to use than the standard scalars of vector fields. We still stress a few differences. What are usual called "scalars" are here distinguished between 0-form and 3-form in a very similar way as we

[1] More precisely an "oriented submanifold". The integration on path from A to B or from B to A have a different sign. Similarly the integration of a flow through a surface going inward or outward have a different sign.

distinguish intensive and extensive properties. And what are usual called "vector fields" are distinguished between 1-form and 2-form. It is still interesting to make such a difference at the mathematical level since they would not behave the same way with a change of coordinate. A fact that stills appears in the system of units.

In $\mathbb{R}^4$ the dimension of the $k-$form $\Lambda^k(\mathbb{R}^3)$ are respectively $1, 4, 6, 4$, and 1. The 0-forms and 4-forms are the usual scalars while 1-forms and 3-forms are the usual 4-vectors. The 2-form are less commun but correspond to anti-symmetric 2-tensors. Generally speaking anti-symmetric properties of tensor are related to integration so have some geometric meaning. Here are some examples of physical quantities written as exterior forms in 4-dimensional.

dimension	basis	quantity	unit
0-fom	1	(same as in $\mathbb{R}^3$)	$*$
1-form	$dx, dy, dz,$ dt	4-gradient $(\partial_t f, \nabla f)$	$*.\boldsymbol{m^{-1}}$ or $*.\boldsymbol{s^{-1}}$
2-form	$dx \wedge dy, \cdots$ $dt \wedge dx, \cdots$	Electromagnetic Field $F = (E, B)$	$*.\boldsymbol{m^{-2}}$ or $*.\boldsymbol{m^{-1}s^{-1}}$
3-form	$dx \wedge dy \wedge dz$,... $dt \wedge dx \wedge dy$,...	Density and flow $J = (\rho, j)$	$*.\boldsymbol{m}^{-3}$ or $*.\boldsymbol{m}^{-2}\boldsymbol{s}^{-1}$
4-form	$dt \wedge dx \wedge dy \wedge dz$	Field Lagrangian	$*.\boldsymbol{m}^{-3}\boldsymbol{s}^{-1}$

For example integrating $J = j\, dt \wedge dx \wedge dy$ would give the amount of matter that has been through a (horizontal) surface within a time interval.

1.3 Tangent vector fields

"Vector field" is also usually used to design another mathematical object that to avoid ambiguity we will call here *tangent vector fields.* If exterior forms are associated with integration, one intead should think of tangent vector fields [LCC+09] informally as *what* describes a flow, ie a transport.

Definition 2. A flow is a familly of application on Ω

$$\phi_t : \Omega \to \Omega \quad t \in \mathbb{R}_+.$$

We should stress here that ϕ_t is well defined at a geometrical level. And even if it looks abstract, it is very simple and natural because it is nothing but the description of how the system has evolved at different times. Moreover if we assume that is a semi-group

$$\phi_{t_1+t_2} = \phi_{t_1} \circ \phi_{t_2} \quad t_1, t_2 \in \mathbb{R}_+$$

one can introduce a tangent vector field such the flow is the solution of a differential equation.

Definition 3. For a flow ϕ_t the associated tangent vector field X is such that for any for point $x_0 \in \Omega$,

$$x(t) = \phi_t(x_0) \quad \Leftrightarrow \quad \begin{cases} x(0) = x_0 \\ \partial_t x(t) = X(x(t)). \end{cases} \tag{1}$$

One could see the tangent vector field as the derivative of the flow "$X = \frac{d}{dt}\phi_t$". So Equation (1) can be thought as the usual way to measure a velocity flow which is to observe the system at different time and “differentiate” this evolution. The main message here is that a tangent vector field it is a completely different mathematical object than the exterior forms. It is used to describe different physical notions and has different basis notation and system of units as for example :

object	basis	quantity	unit
tangent vector	$\partial_x, \partial_y, \partial_z$	velocity of a fluid	$\boldsymbol{m}.s^{-1}$

As illustrated by the system of units tangent vector fields are usually refered as covariant tensors while exterior form are contravariant tensors. In $\mathbb{R}^4$, we also consider transport in the time direction and use the basis element ∂_t. For example one can consider a motionless object but still think of it as transported from the past toward the future. In general we see the evolution from a point (t_0, x_0, y_0, z_0) to another point (t_1, x_1, y_1, z_1) as a transport both in time and space.

1.4 Transported and enlarged submanifold

Tangent vector fields allow us to define a few opérations on a submanifold $\mathcal{V} \subset \Omega$ which are, we recall, our basic object of interest.

Definition 4. With X a tangent vector field associated to a semi-group ϕ_t and $\mathcal{V}$ a submanifold, we define

- An transported submanifold
$$\mathcal{V}(t) = \phi_t(\mathcal{V}) = \{x(t),\, x_0 \in \mathcal{V}\}$$
with $x(t)$ solution of (1).
- An enlarged submanifold
$$\mathcal{I}_t(\mathcal{V}) = \bigcup_{s\in[0,t]} \phi_s(\mathcal{V}) = \{x(s),\, x_0 \in \mathcal{V},\, s \in [0,t]\}$$
that is a submanifold which is one dimensional larger than $\mathcal{V}$.

For example if one think of X describing the velocity field of flow of water, then a transported submanifold would be used to describe anything that is dived in the water and then carried away by the flow. For the enlarged submanifold one can see it as keeping track of the successive positions of transported manifold along the flow as in an numerical simulation. For example in the case of a point $\mathcal{V} = \{x_0\}$, this is just the line traced by the trajectory $x(t)$. In the case of the leave it is the volume in the water that has been covered by the leave on its way.

Both $\mathcal{V}(t)$ and $\mathcal{I}_t(\mathcal{V})$ are natural objects on which integrate an exterior form. This is will be presented in the next section.

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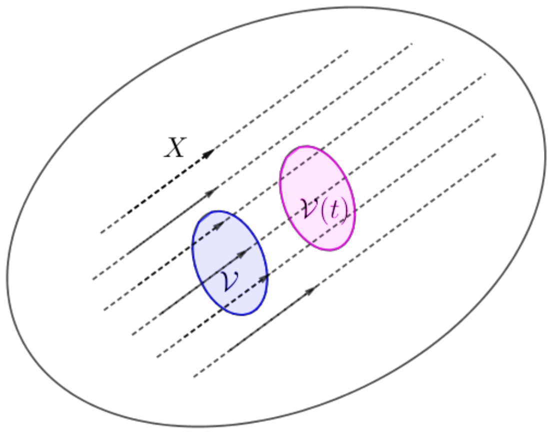

Figure 2: The transported submanifold $\mathcal{V}(t)$.

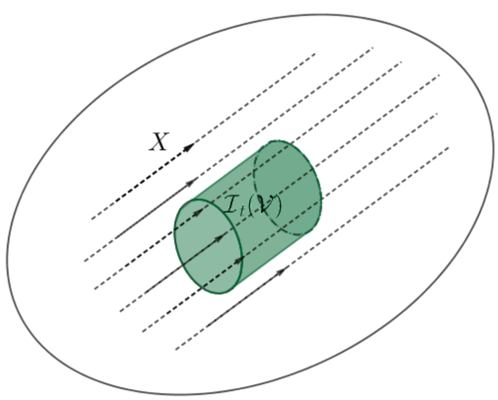

Figure 3: The enlarged submanifold $\mathcal{V}(t)$.

1.5 Pullback transport, Lie derivative and interior product

We now present some basic operations to manipulate exterior form as appears in math textbooks [JJ08, LCC+09, Fra11]. All these operations also have a simple definition in a coordinate system but the goal of these notes is to stay at a geometric point of view and to give an motivation to introduce these objects. Also it makes it clear that they are independent of the choice of the coordinate and therefore should have some physical meaning.

Definition 5. With X a tangent vector field associated to a semi-group ϕ_t and α a k form we introduce the following quantities

- The *pullback transport* $\phi_t^*(\alpha)$ defined as a k form such that for any k dimensional submanifold $\mathcal{V}$

$$[\phi_t^*(\alpha)](\mathcal{V}) = \alpha(\mathcal{V}(t)).$$

- The *Lie derivative* $L_X\alpha$ defined as

$$L_X\alpha = \frac{d}{dt}|_{t=0}\phi_t^*(\alpha)$$

Remark that the Lie derivative is then defined as the k form such that for any k dimensional submanifold $\mathcal{V}$

$$[L_X\alpha](\mathcal{V}) = \lim_{t\to 0}\frac{\alpha(\mathcal{V}(t)) - \alpha(\mathcal{V}(0))}{t}.$$

Definition 6. With X a tangent vector field associated to a semi-group ϕ_t and α a k form we introduce the following quantities

- A $k-1$ form $I_t\alpha$ defined such that for any $k-1$ dimensional submanifold $\mathcal{V}$

$$[I_t\alpha](\mathcal{V}) = \alpha(\mathcal{I}_t(\mathcal{V}))$$

  with $\mathcal{I}_t(\mathcal{V})$ the enlarged submanifold.

- The *interior product* $i_X\alpha$ *defined as*

$$i_X\alpha = \frac{d}{dt}|_{t=0}I_t\alpha$$

That is the $k-1$ form such that for any $k-1$ dimensional submanifold $\mathcal{V}$

$$[i_X\alpha](\mathcal{V}) = \lim_{t\to 0}\frac{\alpha(\mathcal{I}_t(\mathcal{V}))}{t}.$$

This has a natural physical meaning : for example with ρ a density of matter (a 3-form) and X a velocity field, $i_X\rho$ (a 2-form) describes the density flow (For any surface S, $i_X\rho(S)$ is the flow of matter that is going through S).

The approach presented here is actually very general : any operation on submanifolds can be translated into an operation on forms. Indeed considering an application that from a k submanifold $\mathcal{V}$ gives another submanifold $\mathcal{V} \to \tilde{\mathcal{V}}$. We can define an application that from a k form α gives another form $\alpha \to \tilde{\alpha}$ such that

$$\tilde{\alpha}(\tilde{\mathcal{V}}) = \alpha(\mathcal{V}) \tag{2}$$

Equation (2) can also be used to have "dual" application $\tilde{\alpha} \to \alpha$ that from a form $\tilde{\alpha}$ define a k form α.

Question. *If we restrict ourself to the use of this operators, what physics equations could we write ?*

We will work on that question later but first we have to introduce the exterior derivative.

2 Exterior Derivative

2.1 Integration on the boundary

The usual formal definition of the exterior derivative use the derivation at the coordinate level [JJ08, LCC+09, Fra11] but as before it is not very clear at first sight why such an objet would be interesting in physics. So here again is another definition that is informal but that looks very natural and capture the main interest of the object.

Definition 7. For a $k-$form α, the exterior derivative $d\alpha$ is defined as the $(k+1)-$form such that for any $(k+1)$ dimensional submanifold $\mathcal{V}$

$$d\alpha(\mathcal{V}) = \alpha(\partial\mathcal{V}).$$

This is of course the very famous Stokes Theorem.

$$\int_{\mathcal{V}} d\alpha = \int_{\partial\mathcal{V}} \alpha$$

but as in Definition 1 such an equation could also be seen as a definition of an application on the $(k+1)-$submanifolds, that is a $(k+1)-$form and such a definition does not depend on a choice coordinate.

Also boundaries of submanifold are very commun objects : the extremal points of a path, the circle surounding a disc, the surface around a volume. One can also think of the initial and final configurations of a system as the 3 dimension boundaries of the evolving system in time and space seen as a 4 dimension submanifold.

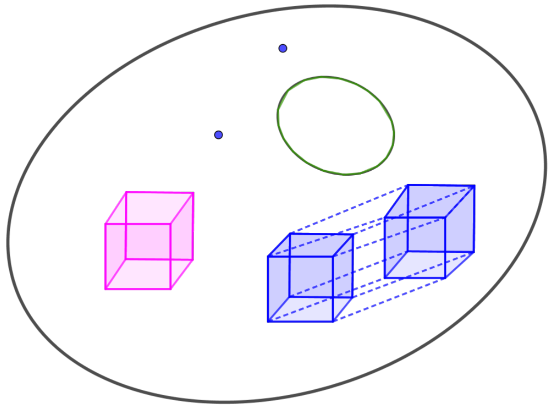

Figure 4: Boundaries of the submanifolds in Figure 1.

In a 3 dimension space whether the form α is a $0, 1$ or 2-form, the exterior derivative $d\alpha$ is called gradient, curl and divergence[2].

$$\Lambda^0(\mathbb{R}^3) \xrightarrow{\text{grad}} \Lambda^1(\mathbb{R}^3) \xrightarrow{\text{curl}} \Lambda^2(\mathbb{R}^3) \xrightarrow{\text{div}} \Lambda^3(\mathbb{R}^3).$$

where $\Lambda^k(\mathbb{R}^3)$ denote the space of $k-$form.

2.2 Conserved quantity

With the previous section in $\mathbb{R}^3$ a divergence-free vector field is then a 2-form α such that $d\alpha = 0$. In $\mathbb{R}^4$ we can propose a similar definition of a conserved quantity. If we go back to our example of evolving system in time and space. For conservation we ask that the integration on the initial and final configurations gives the same result. This is guaranty if $d\alpha = 0$ so we can propose the following definition.

Definition 8. A conserved quantity is a $3-$form[3] J such that $dJ = 0$.

Writting the 3-form in a coordinate frame $J = (\rho, j_x, j_y, j_z)$, the condition $dJ = 0$ is just the Continuous Equation [Fla63]

$$\partial_t \rho + \text{div}(j) = 0.$$

An advantage of the above formulation is that it does not depends on the coordinate frame or the metric. Equivalently for any $\mathcal{V}$ we have $J(\partial\mathcal{V}) = 0$. In word : "What is going into $\mathcal{V}$ is the same as what is going out of $\mathcal{V}$" which is also a very natural non mathematical definition of what a conserved quantity is.

[2]The most commun convention use the metric tensor to transform these forms into vector field. See Section 3.2.

[3]And more generaly a $(n-1)-$form if Ω is n dimensional

2.3 Gauge Invariance

We also state the following important observation : «the boundary of a manifold has no boundary» $\partial(\partial\mathcal{V}) = \emptyset$. Therefore for any form α, we have

$$(d \circ d\alpha)(\mathcal{V}) = \alpha(\partial(\partial\mathcal{V})) = 0$$

and we write the following proposition.

Proposition 9. $d \circ d = 0$

For example in $\mathbb{R}^3$ this is the well known curl $\circ$ grad $= 0$ and div $\circ$ curl $= 0$.

Corollary 10. *(Gauge invariance) For a k-form α and any $(k-1)$ form β we have*

$$d(\alpha + d\beta) = d\alpha.$$

If the physical quantity of interest is $d\alpha$, then the k-form α is not a unique and any $\alpha + d\beta$ works as well. Setting a particular β is to make a Gauge choice. For example for the magnetic field $B = \mathrm{curl}(A + \mathrm{grad}(f))$.

2.4 Cartan's magic formula

We go back to the math textbooks [JJ08, LCC+09, Fra11]. Here are some remarks related to the boundaries (see for example Figure 2) : The boundary of the transported submanifold is just the transport of the boundary of the inital submanifold:

$$\partial[\mathcal{V}(t)] = [\partial\mathcal{V}](t).$$

As a consequence we have this Proposition.

Proposition 11. *With X a tangent vector field associated to a flow ϕ_t and α a k form we have*

$$\phi_t^*(d\alpha) = d\phi_t^*(\alpha) \quad \textit{and} \quad dL_X\alpha = L_X(d\alpha).$$

We also remark following. The boundary of $\mathcal{I}_t(\mathcal{V})$ has three terms : $\mathcal{V}$, $\mathcal{V}(t)$ and the union of $\partial\mathcal{V}(s)$ for $s \in [0, t]$. The latter also corresponds to $\mathcal{I}_t(\partial\mathcal{V})$. Therefore we have[4]

$$[I_t d\beta](\mathcal{V}) = d\beta(\mathcal{I}_t(\mathcal{V})) = \beta(\mathcal{V}(t)) - \beta(\mathcal{V}(0)) - \beta(\mathcal{I}_t(\partial\mathcal{V}))$$

and then obtain the relation known as Cartan's magic formula.

Proposition 12. *(Cartan's magic formula) With X a tangent vector field associated to a flow ϕ_t and α a k form we have*

$$I_t d\alpha = \phi_t^*(\alpha) - \alpha - dI_t\alpha \quad \textit{and} \quad L_X = d \circ i_X + i_X \circ d.$$

[4]The orientation of the boundary parts is a bit tricky.

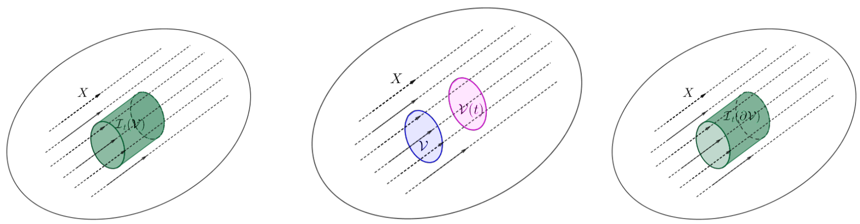

Figure 5: A "Geometric proof" of Cartan's formula $i_X \circ d = L_X - d \circ i_X$ seen as a consequence of $\partial[\mathcal{I}_t(\mathcal{V})] = \mathcal{V}(t) \cup \mathcal{V}(0) \cup \mathcal{I}_t(\partial\mathcal{V})$: $i_X \circ d$ is associated to the figure on the left, L_X to the figure in the center and $d \circ i_X$ to the figure on the right.

2.5 Lie Derivative and forces applied at the boundary

To consider the total force applied on a submanifold $\mathcal{V}$ one would simply write $\overrightarrow{\boldsymbol{F}} = \int_{\mathcal{V}} \overrightarrow{F}$. Unfortunatly if $\mathcal{V} \subset \Omega$ a general manifold ($\neq \mathbb{R}^d$) one can't just sum vectors that are defined at different points and the last equation does'nt make any sense. To overcome this difficulty we will consider a displacement described by a tangent vector field X and consider the work generated by the force as the displacement is made. We call the quantity $F_X = \int_{\mathcal{V}} \overrightarrow{F}.\overrightarrow{X}$ the force in X-direction. For example in the case $\Omega = \mathbb{R}^3$ and a constant vector field $\overrightarrow{X} = \overrightarrow{e_x}, \overrightarrow{e_y}$ or $\overrightarrow{e_z}$ (that describe a translation) one recover the standard definition $\overrightarrow{\boldsymbol{F}} = (F_x, F_y, F_z)$ where $F_x = \int_{\mathcal{V}} \overrightarrow{F}.\overrightarrow{e_x}$. But here F_X is a very general notion because X can then be any tangent vector field. For example in the case of X are the rotations it gives the moment of force. And more generally it could be the force associated to any deformation as for example a dilatation or a contraction. This is the quantity can be generalized in the exterior form formalizm.

Definition 13. If α describes the energy of a system and X is a tangent vector field then $L_X\alpha$ is the associated force in the X-direction.

We then have a 0,1,2 or 3 form whether it is a ponctual force, a linear force, a surface force or a volume force.

Here is the motivation for such a definition. If $\mathcal{V}$ is submanifold that describe a physical object and $\alpha(\mathcal{V})$ is the energy of the system. The tangent vector field X defined a displacememt of the object as the transported manifold $\mathcal{V}(s)$. Then the variation of the energy is given by

$$-\frac{d}{ds}\alpha(\mathcal{V}(s)) = \frac{d}{ds}[\phi_s^*\alpha](\mathcal{V}) = (L_X\alpha)(\mathcal{V}).$$

Remark 14. If $d\alpha = 0$ then $(L_X\alpha)(\mathcal{V}) = (i_X\alpha)(\partial\mathcal{V})$.[5]

We mention here a few examples.

[5] Cartan formula

- Archimedes principle : For the pressure which is a 3−form P that describes the energy per volume. Then the force on a volume $\mathcal{V}$ is $L_X P(\mathcal{V})$ which may correspond to the weight of the fluid and is also given by $(i_X P)(\partial \mathcal{V})$ which is pressure force integrated at the surface.

- Magnetic force : For the magnetic field B is a 2-form and a surface S for which $\partial \mathcal{S}$ is a closed loop with a circulating electric courant i_0, the energy is given by $i_0 B(\mathcal{S})$ and then the force is $i_0 L_X B(\mathcal{S})$ which is the variation of the magnetic flux. But it is also given by $i_0(i_X B)(\partial \mathcal{S})$ which in a coordinate system reads $i_0 \oint_{\partial \mathcal{S}} \overrightarrow{X}.\overrightarrow{B} \times \overrightarrow{d\ell}$ (Lorentz force).

3 The metric and Maxwell equations

3.1 The volume form

Surprisingly so far we only have a manifold Ω and didn't attached any metric on it[6]. But of course it is of fundamental importance in physics. Formally a metric symetric 2-tensor g_{ij} so it is neither a form nor a tangent vector. The first use of the metric for differential form is the definition of a "volume form" ν.

Definition 15. A metric g locally gives a orthonormal basis $dx_1, dx_2, \cdots, dx_n$ and we define

$$\nu = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n.$$

For example once you have defined the meter and what orthogonal means then you can construct the meter-cube. And then have a 3-form ν such that for any 3 dimensional submanifold $\mathcal{V}$

$$\nu(\mathcal{V}) = \text{"Volume of } \mathcal{V} \text{ in } m^3\text{"}.$$

There may be some misleading denomination here. For example with a density ρ that is a 3-form, we write $\rho = \rho(x) dx \wedge dy \wedge dz = \rho(x)\nu$. Here $\rho(x)$ is a 0-form but then when we talk about the density it is not clear if we mean the 0-form or the 3-form. In the general case when $dx_1, \cdots, dx_n$ is not an orthonormal basis $\nu = \sqrt{\det g}\, dx_1 \wedge \cdots \wedge dx_n$.

The volume form also gives a natural way to construct a $(n-1)$-form from a tangent vector field X using the interior product $X \to i_X \nu$ (and then more misleading denomination).

3.2 The $\star$ operator

From a purely mathematical point of view of the exterior forms the metric also appears in a quite indirect way. Remark that the space of k-form has the same dimension as the space of $(n-k)$ form. They are therefore isomorphic. The g

[6] For mathematicians we use differential geometry but not Riemanian geometry

metric gives a method to construct such an isomorphism that is called the Hodge $\star$ operator. Again a metric g locally gives a orthonormal basis $dx_1, dx_2, \cdots, dx_n$ and to give a very simplified definition.

Definition 16. That $\star$-operator associates a $k-$form to a $(n-k)$-form that is the "complement" in the basis[7] :

$$\star\left(\bigwedge_{i\in J} dx_i\right) = \pm \bigwedge_{i\notin J} dx_i \quad \text{for } J \subset \{1, \cdots, n\} \text{ with k element}$$

For example : $\star(dx) = dy \wedge dz \wedge dt$ and $\star(dx \wedge dt) = dy \wedge dz$. We then have a lot of isomorphisms

$$\begin{array}{ccccc} \text{1-form} & & (n-1)\text{-form} & & \text{tangent vector} \\ \alpha & \leftrightarrow & \star\alpha = i_X\nu & \leftrightarrow & X \end{array}$$

It is very commun to make no difference between these objects and use the same name of the three of them. In practice the nature of the object depends on the application. For example in the formula $f(y) - f(x) = \int_x^y \text{grad}(f)$, $\text{grad}(f)$ refer to the 1-form df. But in a gradient descent $\dot{x} = -\text{grad}(f(x))$, or in Newton second law with a potential, $\text{grad}(f)$ refer to the associated tangent vector. These kind of operations occures a lot in relativity where we spend our time moving up and down indices such as $J^\mu \to J_\mu$ or $F^{\mu\nu} \to F_{\mu\nu}$. But they can be completly transparent when g_{ij} is written as the identity matrix. For a simple example in the case of a general g, with a 0-form f we have $\star f = f\nu = f(x)\sqrt{\det g}\, dx_1 \wedge \cdots \wedge dx_n$ and for a n-form $\rho = \rho(x) dx_1 \wedge \cdots \wedge dx_n$ we have $\star\rho = \frac{1}{\sqrt{\det g}}\rho(x)$ (a 0 form).

We now define the Laplace-De Rham Operator.

Definition 17. We define the following operators :

- The operator ∂ that from a k-form gives an $(k-1)$-form such as

$$\partial = \star \circ d \circ \star.$$

- The Laplace-De Rham operator Δ that from a k-form gives an k-form

$$\Delta = \partial \circ d + d \circ \partial.$$

In $\mathbb{R}^3$ and for a $0-$form f (or 3-form), because $\partial f = 0$ this is the usual formula

$$\Delta f = \text{div}(\text{grad}(f))$$

[7] The formula is just wrong because we do not precise the sign which depend on the order of the dx_i and on the signature of g. But from the beginning we didn't precise the orientation of the submanifold neither... We refer to the previously mentioned textbook for a correct definition.

and for a 1-form u (or 2-form), we have the well known[8]

$$\Delta u = \text{grad}(\text{div}(u)) - \text{curl}(\text{curl}(u)).$$

For a general manifold Ω with metric g and a $0-$form f, $\Delta f = \star d \star df$ which reads

$$\Delta f = \frac{1}{\sqrt{\text{det} g}} \sum_{ij} \partial_j (\sqrt{\text{det} g} g^{ij} \partial_i f).$$

In the Minkovski space $\mathbb{R}^{1,3}$, because g is not positive we obtain d'Alembert operator and denote $\Box = \partial \circ d + d \circ \partial$ instead. In the case $d\alpha = 0$ (or $\partial \alpha = 0$) we may also denote $\Box \alpha = \star d \star \alpha$ (or $\Box \alpha = d \star d\alpha$).

3.3 Maxwell equations

One of the most beautiful application of the exterior form formalism is that it gives a clean and unified picture of the classical theory of electromagnetism [Fla63, Fra11]. Here are the Maxwell equations written with differential forms where we drop the constant ϵ_0, μ_0, c. Maxwell Theory is given by the wonderful table

$$\begin{array}{ccccccccc}
\Lambda^0(\mathbb{R}^4) & \overset{d}{\underset{\partial}{\rightleftharpoons}} & \Lambda^1(\mathbb{R}^4) & \overset{d}{\underset{\partial}{\rightleftharpoons}} & \Lambda^2(\mathbb{R}^4) & \overset{d}{\underset{\partial}{\rightleftharpoons}} & \Lambda^3(\mathbb{R}^4) & \overset{d}{\underset{\partial}{\rightleftharpoons}} & \Lambda^4(\mathbb{R}^4) \\
f & \overset{(1)}{\underset{(2)}{\rightleftharpoons}} & (V, A) & \overset{(3)}{\longrightarrow} & (E, B) & \overset{(4)}{\longrightarrow} & 0 & & \\
 & & & & \star(E, B) & \underset{(5)}{\longrightarrow} & (\rho, j) & \underset{(6)}{\longrightarrow} & 0
\end{array}$$

Notice that the dimension of $\Lambda^1(\mathbb{R}^4)$, $\Lambda^2(\mathbb{R}^4)$ and $\Lambda^3(\mathbb{R}^4)$ are 4, 6 and 4. We denote here

- $A^\mu = (V, A)$: the potential and potential vector (usually a $4-$vector). This is a $1-$form as it is linked to the phase of the wave function of the electron (as for example in the Aharonov Bohm effect).
- $F = (E, B)$: the electromagnetic field (usually already a antisymmetric $2-$tensor). This is a $2-$form, one would want to integrate the magnetic field on a surface for example.
- $J^\mu = (\rho, j)$: charge density and current (usually a 4-vector). This is a $3-$form, the density is integrate on a volume and the current on a surface times a time intervale.

A great advantage here that everything is at the "geometric" level. There is no choice of parametrisation of the space. No worry about the change of referencial.

Here every arrow of the diagramm is just the exterior derivative. Bellow are their usual meaning:

[8] But seen here as the definition of Δu.

(1) This is a Gauge invariance, one can change

$$V \to V - \frac{\partial f}{\partial t} \quad \text{et} \quad A \to A + \text{grad}(f)$$

without modifying electromagnetic field, indeed $(3) \circ (1) = 0$.

(2) A particular choice of Gauge called Lorentz's Gauge $(2) = 0$ that is

$$\frac{\partial V}{\partial t} + \text{div}(A) = 0.$$

(3) Electromagnetic fields are expressed with the potential and potential vector:

$$E = -\frac{\partial A}{\partial t} - \text{grad}(V) \quad \text{et} \quad B = \text{curl}(A).$$

(4) Here are the Maxwell-Faraday et Maxwell-Thomson equation

$$\frac{\partial B}{\partial t} + \text{curl}(E) = 0 \quad \text{et} \quad \text{div}(B) = 0$$

This follows of course from $(4) \circ (3) = 0$.

(5) Here are now Maxwell-Gauss and Maxwell-Ampere equations

$$\text{div}(E) = \rho \quad \text{et} \quad -\frac{\partial E}{\partial t} + \text{curl}(B) = j$$

(6) This is the conservation law of the charge

$$\frac{\partial \rho}{\partial t} + \text{div}(j) = 0$$

Again that is $(6) \circ (5) = 0$.

We can finish by a small historical remark, in 1865 the fantastic idea of Maxwell were to notice that $(6) \circ (5) \neq 0$ with Ampere equation as stated at that time. He modified the equation adding the term $\frac{\partial E}{\partial t}$ to obtain a coherent theory. Therefore it is indeed a differential geometry approach that gave nowday's classical electromagnetic theory.

4 De Rham (Trivial) Cohomologie

We have seen in Section 2 that if a form α can be written as $\alpha = d\beta$ it satisfies $d\alpha = 0$. A natural question would be to asked whether the converse is always true. The problem gives rize to a whole domain of study called De Rham Cohomology which happens to be one of the most powerful tools to study and charaterize topological object in differential geometry and algebraic topology. Here we do not go deep into it but just mention one of its simplest result [God70].

Proposition 18. *(De Rham (trivial) Cohomology) In $\mathbb{R}^n$, for any form α that is not a constant $0-$form*

$$d\alpha = 0 \quad \Leftrightarrow \quad \text{there exists } \beta \text{ such that } \alpha = d\beta.$$

This proposition is not true for more complicated topological space and appends to be a very interesting mathematical question. But in $\mathbb{R}^3$ all this is very well known by any undergrad student :

- α is a 0-form : $\text{grad}(\alpha) = 0$ iff there exists $c \in \mathbb{R}$ constant such that $\alpha = c$,
- α is a 1-form : $\text{curl}(\alpha) = 0$ iff there exists a $0-$form β such that $\alpha = \text{grad}(\beta)$,
- α is a 2-form : $\text{div}(\alpha) = 0$ iff there exists a $1-$form β such that $\alpha = \text{curl}(\beta)$,
- α is a 3-form : The equation $\text{div}(\beta) = \alpha$ always has a solution.

In higher dimension, this also implies important properties.

Corollary 19. *If J is a conserved quantity, then there exists β such that $d\beta = J$.*

In particular in $\mathbb{R} \times \mathbb{R}^3$ for any (ρ, j) that satisfies the continuous equation there exists a field $\beta = (\mathcal{E}, \mathcal{B})$ for which Maxwell-Gauss and Maxwell-Ampere equations are valid

$$\text{div}(\mathcal{E}) = \rho \quad \text{and} \quad -\frac{\partial \mathcal{E}}{\partial t} + \text{curl}(\mathcal{B}) = j.$$

This is a purely mathematical construction. This field should be seen as an equivalent to the potential flow that describes an irrotational velocity field as a gradien in hydrodynamics.

Also such a β is not unique and one can made a choice of Gauge. In the case $d(\star\beta) = 0$ we have the following

Corollary 20. *If J is a conserved quantity, then there exists α such that*

$$d \star d\alpha = J.$$

In a coordinate system it is the usual propagation equation with a source

$$\Box \alpha = J \tag{3}$$

We should stress that this is true for *any* conserved quantity with no more information about the physics of the system. However there are important examples where α appears in theorical model or is indeed a real physical quantity. For example :

- The electric charge is conserved. We have the electromagnetic field : (3) with $J = (\rho, j)$ the charge density and current and $\alpha = (V, A)$ potential and vector potential.

So we can ask the following.

Question 21. *What is the associated α for the conservation of energy, momentum, moment of inertia, weak charge,... ?*

5 Euler Lagrange Equations

Here we write the Lagrangian approach for a classical field theory on $\mathbb{R}^n$ using exterior forms.

5.1 Euler Lagrange Equations

As a most simple model the Lagrangian $\mathcal{L}$ is a n-form[9] and we assume the following.

Assumption 22. $\mathcal{L} = \mathcal{L}(\alpha, d\alpha)$ *only depends only on a* $k-$*form* α *and its exterior derivative* $d\alpha$*.*

For $k \geq 1$, the last hypothesis is more restrictive than considering all the derivatives $(\frac{\partial \alpha}{\partial x_i})_{1\leq i\leq n}$ but is also reasonable if we believe that $d\alpha$ has more geometric or physical meaning[10]. Also a nice aspect of this formalism is that it gives a justification why the second order derivatives $\frac{\partial^2 \alpha}{\partial x_i \partial x_j}$ are not used in the Lagrangian : indeed $d \circ d\alpha = 0$.

We write $\mathcal{L}_\alpha := \partial_1 \mathcal{L}(\alpha, d\alpha)$ that is an $(n-k)$-form and $\mathcal{L}_{d\alpha} := \partial_2 \mathcal{L}(\alpha, d\alpha)$ that is a $(n-k-1)$-form such that for a small perturbation $\delta\alpha$ at first order we have

$$\mathcal{L}(\alpha + \delta\alpha, d\alpha + d\delta\alpha) \approx \mathcal{L}(\alpha, d\alpha) + \delta\alpha \wedge \mathcal{L}_\alpha + (d\delta\alpha) \wedge \mathcal{L}_{d\alpha}$$

With this formulation we also assume that $\mathcal{L}$ depends only locally on α, in the sense that if $\delta\alpha$ is supported on a region $\Omega' \subset \mathbb{R}^n$ then $\mathcal{L}$ is not modified outside of Ω'. We now write down the Euler-Lagrange Equation. Remark that we have

$$\mathcal{L}(\alpha + \delta\alpha, d\alpha + d\delta\alpha) \approx \mathcal{L}(\alpha, d\alpha) + \delta\alpha \wedge (\mathcal{L}_\alpha - (-1)^k d\mathcal{L}_{d\alpha}) + d(\delta\alpha \wedge \mathcal{L}_{d\alpha}).$$

For a submanifold $\mathcal{V}$, in order to maximise $\int_{\mathcal{V}} \mathcal{L}(\alpha, d\alpha)$ with fixed boundary conditions on $\partial\mathcal{V}$ for α we obtain the following.

Definition 23. (Euler Lagrange Equation)

$$\mathcal{L}_\alpha - (-1)^k d\mathcal{L}_{d\alpha} = 0$$

For α a 0-form this is the usual Euler Lagrange Equation in Field Theory

$$\frac{\partial \mathcal{L}}{\partial \alpha} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \alpha)} \right).$$

A nice example with α a 1-form is of course the electromagnetic field with $\alpha = (V, A)$ and $J = (\rho, j)$ we have

$$\mathcal{L} = \alpha \wedge J - \frac{1}{2}(d\alpha \wedge \star d\alpha), \quad \mathcal{L}_\alpha = J \quad \text{and} \quad \mathcal{L}_{d\alpha} = \star d\alpha.$$

and then Euler Lagrange Equation gives again the Maxwell equations :

$$d \star d\alpha = J$$

[9] Most commun conventions call the Lagrangian $\star\mathcal{L}$ which is then a 0-form (a scalar).

[10] If α is a 0-form then considering $d\alpha$ is similar to considering all the derivatives $(\frac{\partial \alpha}{\partial x_i})_{1\leq i\leq n}$.

5.2 A conserved quantity in the 1-Form case

Remark that Euler Lagrange Equation directly implies that $d\mathcal{L}_\alpha = 0$. So in particular we have the following.

Corollary 24. *If α is a 1-form then $\mathcal{L}_\alpha$ is a conserved quantity and $\mathcal{L}_{d\alpha}$ is an associated field*[11].

One way to understand Corollary 24 is that assumption 22 add a Gauge symmetry to the system. For example with a perturbation of the form $\delta\alpha = d(\delta f)$ we have $d(\delta\alpha) = 0$ and then

$$\mathcal{L}(\alpha + d\delta f, d\alpha) \approx \mathcal{L}(\alpha, d\alpha) - \delta f \wedge d\mathcal{L}_\alpha + d(\delta f \wedge \mathcal{L}_\alpha).$$

and finally obtain $d\mathcal{L}_\alpha = 0$ as for the Euler Lagrange Equation. Again we have a nice example with the electromagnetic field where $\mathcal{L}_\alpha = J$ and $\mathcal{L}_{d\alpha} = \star d\alpha$.

5.3 Noether Theorem

We now mention the famous Noether Theorem. Remark that Euler Lagrange equation implies that

$$\mathcal{L}(\alpha + \delta\alpha, d\alpha + d\delta\alpha) - \mathcal{L}(\alpha, d\alpha) \approx d(\delta\alpha \wedge \mathcal{L}_{d\alpha})$$

We denote ξ an infinitesimal transformation $\alpha \to \alpha + \delta\alpha^\xi$ and $\mathcal{L} \to \mathcal{L} + \delta\mathcal{L}^\xi$. If the system is invariant by this transformation $\delta\mathcal{L}^\xi = 0$ then $d(\delta\alpha^\xi \wedge \mathcal{L}_{d\alpha}) = 0$ so $\delta\alpha^\xi \wedge \mathcal{L}_{d\alpha}$ is a conserved quantity. We can state a more general theorem [Olv93].

Theorem 25. *(Noether Theorem) If $\delta\mathcal{L}^\xi = d(\delta\Lambda^\xi)$ then $\delta\alpha^\xi \wedge \mathcal{L}_{d\alpha} - \delta\Lambda^\xi$ is a conserved quantity.*

A very nice application of Noether Theorem is of course the conservation of energy and momentum.

5.4 The stress-energy tensor

We consider translations of the system and more generaly the transport along a flow given by vector field X. Notice that we have $L_X\mathcal{L} = d(i_X\mathcal{L})$ (Cartan's magic formula) so we can apply Noether Theorem. We compute

$$L_X\mathcal{L} = \mathcal{L} + L_X\alpha \wedge \mathcal{L}_\alpha + (dL_X\alpha) \wedge \mathcal{L}_{d\alpha} + L_X\mathcal{L}|_{\alpha,d\alpha}$$

and then obtain

$$d(L_X\alpha \wedge \mathcal{L}_{d\alpha} - i_X\mathcal{L}) = -L_X\mathcal{L}|_{\alpha,d\alpha}.$$

For example in the case of the electromagnetism Lagrangian and $X = \partial_t$ for translation in time we obtain Poynting's theorem

$$\partial_t(\frac{1}{2}(|E|^2 + |B|^2) + \vec{A}.\vec{j}) + \mathrm{div}(E \times B) = \vec{A}.\partial_t\vec{j}.$$

For more general Lagrangian $\mathcal{L}$ we can define the following.

[11] As in Corollary 19

Definition 26. We call $L_X\alpha \wedge \mathcal{L}_{d\alpha} - i_X\mathcal{L}$ the *stress-energy tensor* if X is a translation.

In a coordinate system and with the translations $X = \partial_\nu$, this is the usual formula for the stress-energy tensor defined from a Lagrangian

$$T_{\mu\nu} = \partial_\nu \alpha^\eta \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \alpha)} \right)_\eta - \eta_{\mu\nu}\mathcal{L}.$$

And we also state the conservation of energy and momentum in a general setting.

Corollary 27. *If $L_X\mathcal{L}|_{\alpha,d\alpha} = 0$, ie. the Lagrangian is invariant by the transformation induced by the vector field X, then $(L_X\alpha) \wedge \mathcal{L}_2 - i_X\mathcal{L}$ is a conserved quantity.*

5.5 Hamilton Equation

The energy tensor

$$\mathcal{H} = L_X\alpha \wedge \mathcal{L}_{d\alpha} - i_X\mathcal{L}.$$

corresponds to the definition of the Hamiltonian in the case $X = \partial_t$. With some computation we obtain for small perturbation

$$\begin{aligned} \delta\mathcal{H} = &-\delta\alpha \wedge L_X\mathcal{L}_2 + L_X\alpha \wedge \delta\mathcal{L}_2 - di_X(\delta\alpha \wedge \mathcal{L}_{d\alpha}) \\ &+ i_X(\delta\alpha \wedge (\mathcal{L}_\alpha - (-1)^k d\mathcal{L}_{d\alpha})) \end{aligned}$$

There if we assume that $\mathcal{H} = \mathcal{H}(\alpha, \mathcal{L}_{d\alpha})$ and that the small is perturbation is given by

$$\delta\mathcal{H} = \delta\alpha \wedge H_1 + \delta\mathcal{L}_{d\alpha} \wedge H_2$$

we obtain the Hamilton Equations.

Proposition 28. *(Hamilton Equations) Euler Lagrange Equations are equivalent to*

$$H_1 = -L_X\mathcal{L}_{d\alpha} \quad \text{and} \quad H_2 = L_X\alpha.$$

Remark 29. The stationary form (for which $L_X\alpha = 0$ and then $L_X\mathcal{L}_2 = 0$) which that are critical point of the Hamiltonian ($\delta H = 0$ for any $\delta\alpha$) solve Euler Lagrange equation. In particular if α is such that it minimizes the energy.

5.6 Gravitational waves (?)

We finish by giving a partial answer for Question 21. One can find in a book on General Relativity this equation used to describe gravitational waves

$$\Box\tilde{h} = T$$

where $\tilde{h}$ is constructed with the perturbation of the metric g around the flat Minkowski metric and T is the 4×4 Stress-Energy tensor. Here one can think of each line of T as a 3-form which corresponds to the conservation of energy (first line) and the conservation of momentum (the three others lines) and then the line of $\tilde{h}$ play the role of the associated α in Corollary 20 and the associated β in Corollary 19 can be interpreted as the classical gravitational field.

Unfortunatly general relativity is much more complicated and the Conservation of energy/momentum is true only at first order when gravity is not too strong.

6 Summary table

$m^{-2}s^{-1}$	$dx \wedge dy \wedge dt$
gradient, curl, divergence	d
irrotational, divengence free, conserved quantity	$d\alpha = 0$
Gauge invariance	$d(\alpha + d\beta) = d\alpha$
De Rham cohomology	$d\alpha = 0 \Rightarrow \alpha = d\beta$?
Maxwell equations	$dF = 0, \quad d \star F = 0$
Propagation wave with source	$d \star d\alpha = J$
Euler Lagrange equation	$\mathcal{L}_\alpha - (-1)^k d\mathcal{L}_{d\alpha} = 0$
Stress-energy tensor	$L_X\alpha \wedge \mathcal{L}_{d\alpha} - i_X\mathcal{L}$

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