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PSEUDODIFFERENTIAL OPERATORS ON DIFFERENTIAL GROUPOIDS

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We construct an algebra of pseudodifferential operators on each groupoid in a class that generalizes differentiable groupoids to allow manifolds with corners. We show that this construction encompasses many examples. The subalgebra of regularizing operators is identified with the smooth algebra of the groupoid, in the sense of non-commutative geometry. Symbol calculus for our algebra lies in the Poisson algebra of functions on the dual of the Lie algebroid of the groupoid. As applications, we give a new proof of the Poincaré-Birkhoff-Witt theorem for Lie algebroids and a concrete quantization of the Lie-Poisson structure on the dual A^* of a Lie algebroid.

Introduction.

Certain important applications of pseudodifferential operators require variants of the original definition. Among the many examples one can find in the literature are regular or adiabatic families of pseudodifferential operators [**2**, **41**], pseudodifferential operators along the leaves of foliations [**6**, **8**, **28**, **29**], on coverings [**9**, **30**] or on certain singular spaces [**21**, **22**, **25**, **26**].

Since these classes of operators share many common features, it is natural to ask whether they can be treated in a unified way. In this paper we shall suggest a positive answer to this question. For any "almost differential" groupoid (a class which allows manifolds with corners), we construct an algebra of pseudodifferential operators. We then show that our construction recovers (almost) all the classes described above (for operators on manifolds with boundary, our algebra is slightly smaller than the one defined in [**21**]). We expect our results to have applications to analysis on singular spaces.

Our construction and results owe a great deal to the previous work of several authors, especially Connes [**6**] and Melrose [**20**, **21**, **23**]. A hint of the direction we take was given at the end of [**38**]. The basic idea of our construction is to consider families of pseudodifferential operators along the fibers of the domain (or source) map of the groupoid. More precisely, for any almost differentiable groupoid (see Definition 3) we consider the fibers $\mathcal{G}_x = d^{-1}(x)$ of the domain map d, which consist of all arrows with domain x. It follows from the definition of an almost differentiable groupoid that these fibers are smooth manifolds (without corners). The calculus of pseudodifferential operators on smooth manifolds is well understood and by now a classical subject, see for example [**14**]. We shall consider differentiable families of pseudodifferential operators P_x on the smooth manifolds $\mathcal{G}_x$. Right translation by $g \in \mathcal{G}$ defines an isomorphism $\mathcal{G}_x \equiv \mathcal{G}_y$ where x is the domain of g and y is the range of g. We say that the family P_x is invariant if P_x transforms to P_y under the diffeomorphisms above (for all g). The algebra $\Psi^\infty(\mathcal{G})$ of *pseudodifferential operators on* $\mathcal{G}$ that we shall consider will consist of invariant differentiable families of operators P_x as explained above (the actual definition also involves a technical condition on the support of these operators). See Definition 7 for details. The relation with the work of Melrose relies on an alternative description of our algebra as an algebra of distributions on $\mathcal{G}$ with suitable properties (compactly supported, conormal, and with singular support contained in the set of units). This is contained in Theorem 7. The difference between our theory and Melrose's lies in the fact that he considers a compactification of $\mathcal{G}$ as a manifold with corners, and his distributions are allowed to extend to the compactification, with precise behavior at the boundary. This is useful for the analysis of these operators. In contrast, our work is purely algebraic (or geometric, depending on whether one considers Lie algebroids as part of geometry or algebra).

We now review the contents of the sections of this paper. In the first section we recall the definitions of a groupoid, of a Lie algebroid, and of the less known concept of *local* Lie groupoid. We extend the definition of a Lie groupoid to include manifolds with corners. These groupoids are called *almost differentiable* groupoids. The second section contains the definition of a pseudodifferential operator on a groupoid (really a family of pseudodifferential operators, as explained above) and the proof that they form an algebra, if a support condition is included. We also extend this definition to include local Lie groupoids. This is useful in the third section where we use this to give a new proof of the Poincaré-Birkhoff-Witt theorem for Lie algebroids. In the process of proving this theorem we also exemplify our definition of pseudodifferential operators on an almost differentiable groupoid by describing the differential operators in this class. As an application we give an explicit construction of a deformation quantization of the Lie-Poisson structure on A^*, the dual of Lie algebroid A. The section entitled "Examples" contains just what the title suggests: for many particular examples of groupoids $\mathcal{G}$, we explicitly describe the algebra $\Psi^\infty(\mathcal{G})$ of pseudodifferential operators on $\mathcal{G}$. This recovers classes of operators that were previously defined using *ad hoc* constructions. Our definition is often not only more general, but also simpler. This is the case for operators along the leaves of foliations [**8**, **28**] or adiabatic families of operators. Since one of our main themes is that the Lie algebras of vector fields that are central in [**24**] are in fact the spaces of sections of Lie algebroids, we describe these Lie algebroids explicitly in each of our examples. In the sixth section of the paper, we describe the convolution kernels (called *reduced* kernels) of operators in $\Psi^\infty(\mathcal{G})$. Then we extend to our setting some fundamental results on principal symbols, by reducing to the classical results. This makes our proofs short (and easy). Finally, the last section treats the action of $\Psi^\infty(\mathcal{G})$ on functions on the units of $\mathcal{G}$, and a few related topics.

Recently, we have learned of certain related results by Monthubert, see [**27**] and the references therein. Our paper was circulated as preprint funct-an/9702004. The first author would like to thank Richard Melrose for several useful conversations.

1. Preliminaries.

In the following we allow manifolds to have corners. Thus by "manifold" we shall mean a C^∞ manifold, possibly with corners, and by a "smooth manifold" we shall mean a manifold without corners. By definition, if M is a manifold with corners, then every point $p \in M$ has coordinate neighborhoods diffeomorphic to $[0,\infty)^k \times \mathbb{R}^{n-k}$. The transition functions between such coordinate neighborhoods must be smooth everywhere (including on the boundary). We shall use the following definition of submersions between manifolds (with corners).

Definition 1. A submersion between two manifolds with corners M and N is a differentiable map $f : M \to N$ such that $df_x : T_xM \to T_{f(x)}N$ is onto for any $x \in M$ and such that if $df_x(v)$ is an inward pointing tangent vector to N, then v is an inward pointing tangent vector to M.

The reason for introducing the definition above is that for any submersion $f : M \to N$, the set $M_y = f^{-1}(y)$, $y \in N$ is a *smooth* manifold, just as for submersions of smooth manifolds.

We shall study groupoids endowed with various structures. ([**33**] is a general reference for some of what follows.) We recall first that a small category is a category whose class of morphisms is a set. The class of objects of a small category is then a set as well.

Definition 2. A groupoid is a small category $\mathcal{G}$ in which every morphism is invertible.

This is the shortest but least explicit definition. We are going to make this definition more explicit in cases of interest. The set of objects, or units, of $\mathcal{G}$ will be denoted by

$$M = \mathcal{G}^{(0)} = \mathrm{Ob}(\mathcal{G}).$$

The set of morphisms, or arrows, of $\mathcal{G}$ will be denoted by

$$\mathcal{G}^{(1)} = \mathrm{Mor}(\mathcal{G}).$$

We shall sometimes write $\mathcal{G}$ instead of $\mathcal{G}^{(1)}$ by abuse of notation. For example, when we consider a space of functions on $\mathcal{G}$, we actually mean a space of functions on $\mathcal{G}^{(1)}$. We will denote by $d(g)$ [respectively $r(g)$] the *domain* [respectively, the *range*] of the morphism $g : d(g) \to r(g)$. We thus obtain functions

$$d, r : \mathcal{G}^{(1)} \longrightarrow \mathcal{G}^{(0)} \tag{1}$$

that will play an important role bellow. The multiplication operator $\mu : (g, h) \mapsto \mu(g, h) = gh$ is defined on the set of composable pairs of arrows $\mathcal{G}^{(2)}$:

$$\mu : \mathcal{G}^{(2)} = \mathcal{G}^{(1)} \times_M \mathcal{G}^{(1)} := \{(g, h) : d(g) = r(h)\} \longrightarrow \mathcal{G}^{(1)}. \tag{2}$$

The inversion operation is a bijection $\iota : g \mapsto g^{-1}$ of $\mathcal{G}^{(1)}$. Denoting by $u(x)$ the identity morphism of the object $x \in M = \mathcal{G}^{(0)}$, we obtain an inclusion of $\mathcal{G}^{(0)}$ into $\mathcal{G}^{(1)}$. We see that a groupoid $\mathcal{G}$ is completely determined by the spaces $\mathcal{G}^{(0)}$ and $\mathcal{G}^{(1)}$ and the structural morphisms d, r, μ, u, ι. We sometimes write $\mathcal{G} = (\mathcal{G}^{(0)}, \mathcal{G}^{(1)}, d, r, \mu, u, \iota)$. The structural maps satisfy the following properties:

(i) $r(gh) = r(g)$, $d(gh) = d(h)$ for any pair $(g, h) \in \mathcal{G}^{(2)}$, and the partially defined multiplication μ is associative.

(ii) $d(u(x)) = r(u(x)) = x$, $\forall x \in \mathcal{G}^{(0)}$, $u(r(g))g = g$ and $gu(d(g)) = g$, $\forall g \in \mathcal{G}^{(1)}$ and $u : \mathcal{G}^{(0)} \to \mathcal{G}^{(1)}$ is one-to-one.

(iii) $r(g^{-1}) = d(g)$, $d(g^{-1}) = r(g)$, $gg^{-1} = u(r(g))$ and $g^{-1}g = u(d(g))$.

Definition 3. An almost differentiable groupoid $\mathcal{G} = (\mathcal{G}^{(0)}, \mathcal{G}^{(1)}, d, r, \mu, u, \iota)$ is a groupoid such that $\mathcal{G}^{(0)}$ and $\mathcal{G}^{(1)}$ are manifolds with corners, the structural maps d, r, μ, u, ι are differentiable, and the domain map d is a submersion.

We observe that ι is a diffeomorphism and hence d is a submersion if and only if $r = d \circ \iota$ is a submersion. Also, it follows from the definition that each fiber $\mathcal{G}_x = d^{-1}(x) \subset \mathcal{G}^{(1)}$ is a smooth manifold whose dimension n is constant on each connected component of $\mathcal{G}^{(0)}$. The étale groupoids considered in [5] are extreme examples of differentiable groupoids (corresponding to $\dim \mathcal{G}_x = 0$). If $\mathcal{G}^{(0)}$ is smooth (i.e. if it has no corners), then $\mathcal{G}^{(1)}$ is also smooth and $\mathcal{G}$ becomes what is known as a differentiable, or Lie groupoid.[1]

We now introduce a few important geometric objects associated to an almost differentiable groupoid.

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[1]Earlier terminology, such as in [19], used the term Lie groupoid only for differentiable groupoids in which every pair of objects is connected by a morphism.

The vertical tangent bundle (along the fibers of d) of an almost differentiable groupoid $\mathcal{G}$ is

$$T_d\mathcal{G} = \ker d_* = \bigcup_{x\in\mathcal{G}^{(0)}} T\mathcal{G}_x \subset T\mathcal{G}^{(1)}. \tag{3}$$

Its restriction $A(\mathcal{G}) = T_d\mathcal{G}\big|_{\mathcal{G}^{(0)}}$ to the set of units is the Lie algebroid of $\mathcal{G}$ [**19**, **31**]. We denote by $T_d^*\mathcal{G}$ the dual of $T_d\mathcal{G}$ and by $A^*(\mathcal{G})$ the dual of $A(\mathcal{G})$. In addition to these bundles we shall also consider the bundle Ω_d^λ of λ-densities along the fibers of d. If the fibers of d have dimension n, then $\Omega_d^\lambda = |\Lambda^n T_d^*\mathcal{G}|^\lambda = \cup_x \Omega^\lambda(\mathcal{G}_x)$. By invariance these bundles can be obtained as pull-backs of bundles on $\mathcal{G}^{(0)}$. For example $T_d\mathcal{G} = r^*(A(\mathcal{G}))$ and $\Omega_d^\lambda = r^*(\mathcal{D}^\lambda)$, where $\mathcal{D}^\lambda$ denotes $\Omega_d^\lambda|_{\mathcal{G}^{(0)}}$. If E is a (smooth complex) vector bundle on the set of units $\mathcal{G}^{(0)}$, then the pull-back bundle $r^*(E)$ on $\mathcal{G}$ will have right invariant connections obtained as follows. A connection ∇ on E lifts to a connection on $r^*(E)$. Its restriction to any fiber $\mathcal{G}_x$ defines a linear connection in the usual sense, which is denoted by ∇_x. It is easy to see that these connections are right invariant in the sense that

$$R_g^*\nabla_x = \nabla_y, \quad \forall g \in \mathcal{G} \ \text{ such that } \ r(g) = x \text{ and } d(g) = y. \tag{4}$$

The bundles considered above will thus have invariant connections.

The bundle $A(\mathcal{G})$, called the *Lie algebroid* of $\mathcal{G}$, plays in the theory of almost differentiable groupoids the rôle Lie algebras play in the theory of Lie groups. We recall for the benefit of the reader the definition of a Lie algebroid [**31**].

Definition 4. A Lie algebroid A over a manifold M is a vector bundle A over M together with a Lie algebra structure on the space $\Gamma(A)$ of smooth sections of A, and a bundle map $\rho : A \to TP$, extended to a map between sections of these bundles, such that
(i) $\rho([X, Y]) = [\rho(X), \rho(Y)]$; and
(ii) $[X, fY] = f[X, Y] + (\rho(X)f)Y$
for any smooth sections X and Y of A and any smooth function f on M.

Note that we allow the base M in the definition above to be a manifold with corners.

If $\mathcal{G}$ is an almost differentiable groupoid, then $A(\mathcal{G})$ will naturally have the structure of a Lie algebroid [**19**]. Let us recall how this structure is defined (the original definition easily extends to include manifolds with corners). Clearly $A(\mathcal{G})$ is a vector bundle. The right translation by an arrow $g \in \mathcal{G}$ defines a diffeomorphism $R_g : \mathcal{G}_{r(g)} \ni g' \to g'g \in \mathcal{G}_{d(g)}$. This allows us to talk about right invariant differential geometric quantities as long as they are completely determined by their restriction to all submanifolds $\mathcal{G}_x$. This is true of functions and d–vertical vector fields, and this is all that is needed to define the Lie algebroid structure on $A(\mathcal{G})$. The sections of $A(\mathcal{G})$ are in one-to-one correspondence with vector fields X on $\mathcal{G}$ that are d-vertical, in the sense that $d_*(X(g)) = 0$, and right invariant. The condition $d_*(X(g)) = 0$ means that X is tangent to the submanifolds $\mathcal{G}_x$, the fibers of d. The Lie bracket $[X, Y]$ of two d-vertical right-invariant vector fields X and Y will also be d-vertical and right-invariant, and hence the Lie bracket induces a Lie algebra structure on the sections of $A(\mathcal{G})$. To define the action of the sections of $A(\mathcal{G})$ on functions on $\mathcal{G}^{(0)}$, observe that the right invariance property makes sense also for functions on $\mathcal{G}$ and that $\mathcal{C}^\infty(\mathcal{G}^{(0)})$ may be identified with the subspace of right–invariant functions on $\mathcal{G}$. If X is a right–invariant vector field on $\mathcal{G}$ and f is a right-invariant function on $\mathcal{G}$, then $X(f)$ will still be a right invariant function. This identifies the action of $\Gamma(A(\mathcal{G}))$ on functions on $\mathcal{G}^{(0)}$.

Not every Lie algebroid is the Lie algebroid of a Lie groupoid (see [**1**] for an example). However, every Lie algebroid *is* associated to a *local* Lie groupoid [**32**]. The definition of a local Lie (or more generally, almost differentiable) groupoid [**10**] is obtained by relaxing the condition that the multiplication μ be everywhere defined on $\mathcal{G}^{(2)}$ (see Equation (2)), and replacing it by the condition that μ be defined in a neighborhood $\mathcal{U}$ of the set of units.

Definition 5 (van Est)**.** An almost differentiable local groupoid $\mathcal{L} = (\mathcal{L}^{(0)}, \mathcal{L}^{(1)})$ is a pair of manifolds with corners together with structural morphisms $d, r : \mathcal{L}^{(1)} \to \mathcal{L}^{(0)}$, $\iota : \mathcal{L}^{(1)} \to \mathcal{L}^{(1)}$, $u : \mathcal{L}^{(0)} \to \mathcal{L}^{(1)}$ and $\mu : \mathcal{U} \to \mathcal{L}^{(1)}$, where $\mathcal{U}$ is a neighborhood of $(u \times u)(\mathcal{L}^{(0)}) = \{(u(x), u(x))\}$ in $\mathcal{L}^{(2)} = \{(g, h), d(g) = r(h)\} \subset \mathcal{L}^{(1)} \times \mathcal{L}^{(1)}$. The structural morphisms are required to be differentiable maps such that d is a submersion, u is an embedding, and to satisfy the following properties:

(i) The products $u(d(g))g$, $gu(r(g))$, gg^{-1} and $g^{-1}g$ are defined and coincide with, respectively, g, g, $u(r(g))$ and $u(d(g))$, where we denoted $g^{-1} = \iota(g)$ as usual.

(ii) If gh is defined, then $h^{-1}g^{-1}$ is defined and equal to $(gh)^{-1}$.

(iii) (Local associativity) If gg', $g'g''$, and $(gg')g''$ are defined, then $g(g'g'')$ is also defined and equal to $(gg')g''$.

The set $\mathcal{U}$ is the set of arrows for which the product $gh = \mu(g, h)$ is defined.

We see that the only difference between a groupoid and a local groupoid $\mathcal{L}$ is the fact that the condition $d(g) = r(h)$ is necessary for the product $gh = \mu(g, h)$ to be defined, but not sufficient in general. The product is defined as soon as the arrows g and h are "small enough." A consequence of this definition is that the right multiplication by an arrow $g \in \mathcal{L}^{(1)}$ defines only a diffeomorphism

$$\mathcal{U}_{g^{-1}} \ni g' \to g'g \in \mathcal{U}_g \tag{5}$$

of an open (and possibly empty) subset $\mathcal{U}_{g^{-1}}$ of $\mathcal{L}_y$, $y = r(g)$ to an open subset $\mathcal{U}_g \subset \mathcal{L}_x$, $x = d(g)$. This will not affect the considerations above, however, so we can associate a Lie algebroid $A(\mathcal{L})$ to any almost differentiable local groupoid $\mathcal{L}$.

In the following, when considering groupoids, we shall sometimes refer to them as *global* groupoids, in order to stress the difference between groupoids and local groupoids.

2. Main definition.

Consider a complex vector bundle E on the space of units $\mathcal{G}^{(0)}$ of an almost differentiable groupoid $\mathcal{G}$. Denote by $r^*(E)$ its pull-back to $\mathcal{G}^{(1)}$. Right translations on $\mathcal{G}$ define linear isomorphisms

$$U_g : \mathcal{C}^\infty(\mathcal{G}_{d(g)}, r^*(E)) \to \mathcal{C}^\infty(\mathcal{G}_{r(g)}, r^*(E)) \\ (U_g f)(g') = f(g'g) \in (r^*E)_{g'}, \tag{6}$$

which makes sense because $(r^*E)_{g'} = (r^*E)_{g'g} = E_{r(g')}$.

If $\mathcal{G}$ is merely a *local* groupoid, then (6) is replaced by the isomorphisms

$$U_g : \mathcal{C}^\infty(\mathcal{U}_g, r^*(E)) \to \mathcal{C}^\infty(\mathcal{U}_{g^{-1}}, r^*(E)) \tag{7}$$

defined for the open subsets $\mathcal{U}_g \subset \mathcal{G}_{d(g)}$ and $\mathcal{U}_{g^{-1}} \subset \mathcal{G}_{r(g)}$ defined in (5).

Let $B \subset \mathbb{R}^n$ be an open subset. Define the space $\mathcal{S}^m(B \times \mathbb{R}^n)$ of symbols on the bundle $B \times \mathbb{R}^n \to B$ as in [**14**] to be the set of smooth functions $a : B \times \mathbb{R}^n \to \mathbb{C}$ such that

$$|\partial_y^\alpha \partial_\xi^\beta a(y, \xi)| \leq C_{K,\alpha,\beta}(1 + |\xi|)^{m-|\beta|} \tag{8}$$

for any compact set $K \subset B$ and any multi-indices α and β. An element of one of our spaces S^m should properly be said to have "order less than or equal to m"; however, by abuse of language we will say that it has "order m".

A symbol $a \in \mathcal{S}^m(B \times \mathbb{R}^n)$ is called *classical* if it has an asymptotic expansion as an infinite sum of homogeneous symbols $a \sim \sum_{k=0}^\infty a_{m-k}$, a_l homogeneous of degree l: $a_l(y, t\xi) = t^l a_l(y, \xi)$ if $\|\xi\| \geq 1$ and $t \geq 1$. ("Asymptotic expansion" is used here in the sense that $a - \sum_{k=0}^{N-1} a_{m-k}$ belongs to $\mathcal{S}^{m-N}(B \times \mathbb{R}^n)$.) The space of classical symbols will be denoted by $\mathcal{S}^m_{\text{cl}}(B \times \mathbb{R}^n)$. We shall be working exclusively with classical symbols in this paper.

This definition immediately extends to give spaces $\mathcal{S}^m_{\text{cl}}(E; F)$ of symbols on E with values in F, where $\pi : E \to B$ and $F \to B$ are smooth euclidian vector bundles. These spaces, which are independent of the metrics used in their definition, are sometimes denoted $\mathcal{S}^m_{\text{cl}}(E; \pi^*(F))$. Taking $E = B \times \mathbb{R}^n$ and $F = \mathbb{C}$ one recovers $\mathcal{S}^m_{\text{cl}}(B \times \mathbb{R}^n) = \mathcal{S}^m_{\text{cl}}(B \times \mathbb{R}^n; \mathbb{C})$.

A pseudodifferential operator P on B is a linear map $P : \mathcal{C}_c^\infty(B) \to \mathcal{C}^\infty(B)$ that is locally of the form $P = a(y, D_y)$ plus a regularizing operator, where for any complex valued symbol a on $T^*W = W \times \mathbb{R}^n$, W an open subset of $\mathbb{R}^n$, one defines $a(y, D_y) : \mathcal{C}_c^\infty(W) \to \mathcal{C}^\infty(W)$ by

$$a(y, D_y)u(y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{iy\cdot\xi} a(y, \xi)\hat{u}(\xi)d\xi. \tag{9}$$

Recall that an operator $T : \mathcal{C}_c^\infty(U) \to \mathcal{C}^\infty(V)$ is called *regularizing* if and only if it has a smooth distribution (or Schwartz) kernel. This happens if and only if T is pseudodifferential of order $-\infty$.

The class of a in $\mathcal{S}_{\text{cl}}^m(T^*W)/\mathcal{S}_{\text{cl}}^{m-1}(T^*W)$ does not depend on any choices; the collection of all these classes, for all coordinate neighborhoods W, patches together to define a class $\sigma_m(P) \in \mathcal{S}_{\text{cl}}^m(T^*W)/\mathcal{S}_{\text{cl}}^{m-1}(T^*W)$, which is called *the principal symbol* of P. If the operator P acts on sections of a vector bundle E, then the principal symbol $\sigma_m(P)$ will belong to $\mathcal{S}_{\text{cl}}^m(T^*B; \text{End}(E))/\mathcal{S}_{\text{cl}}^{m-1}(T^*B; \text{End}(E))$. See [**14**] for more details on all these constructions.

We shall sometimes refer to pseudodifferential operators acting on a smooth manifold as *ordinary* pseudodifferential operators, in order to distinguish them from pseudodifferential operators on groupoids, a class of operators, which we now define (and which are really *families* of ordinary pseudodifferential operators).

Throughout this paper, we shall denote by $(P_x, x \in \mathcal{G}^{(0)})$ a family of order m pseudodifferential operators P_x, acting on the spaces $\mathcal{C}_c^\infty(\mathcal{G}_x, r^*(E))$ for some vector bundle E over $\mathcal{G}^{(0)}$. Operators between sections of two *different* vector bundles E_1 and E_2 are obtained by considering $E = E_1 \oplus E_2$.

Definition 6. A family $(P_x, x \in \mathcal{G}^{(0)})$ as above is called differentiable if for any open set $V \subset \mathcal{G}$, diffeomorphic through a fiber preserving diffeomorphism to $d(V) \times W$, for some open subset $W \subset \mathbb{R}^n$, and for any $\phi \in \mathcal{C}_c^\infty(V)$, we can find $a \in \mathcal{S}_{\text{cl}}^m(d(V) \times T^*W; \text{End}(E))$ such that $\phi P_x \phi$ corresponds to $a(x, y, D_y)$ under the diffeomorphism $\mathcal{G}_x \cap V \simeq W$, for each $x \in d(V)$.

A fiber preserving diffeomorphism is a diffeomorphism $\psi : d(V) \times W \to V$ satisfying $d(\psi(x, w)) = x$. Thus we require that the operators P_x be given in local coordinates by symbols a_x that depend smoothly on all variables, in particular, on $x \in \mathcal{G}^{(0)}$.

Definition 7. An order m invariant pseudodifferential operator P on an almost differentiable groupoid $\mathcal{G}$, acting on sections of the vector bundle E, is a differentiable family $(P_x, x \in \mathcal{G}^{(0)})$ of order m classical pseudodifferential operators P_x acting on $\mathcal{C}_c^\infty(\mathcal{G}_x, r^*(E))$ and satisfying

$$P_{r(g)} U_g = U_g P_{d(g)} \text{ (invariance)} \tag{10}$$

for any $g \in \mathcal{G}^{(1)}$, where U_g is as in (6).

Replacing the coefficient bundle E by $E \otimes \mathcal{D}^\lambda$ and using the isomorphism $\Omega_d^\lambda \simeq r^*(\mathcal{D}^\lambda)$, we obtain operators acting on sections of density bundles. Note that P can generally *not* be considered as a single pseudodifferential operator on $\mathcal{G}^{(1)}$. This is because a family of pseudodifferential operators on a smooth manifold M, parametrized by a smooth manifold B, is not a pseudodifferential operator on the product $M \times B$, although it acts naturally on $\mathcal{C}_c^\infty(M \times B)$. (See [**2**] or [**14**], page 94.)

Recall [**13**] that distributions on a manifold Y with coefficients in the bundle E_0 are continuous linear maps $\mathcal{C}_c^\infty(Y, E_0' \otimes \Omega) \to \mathbb{C}$, where E_0' is the dual bundle to E_0 and $\Omega = \Omega(Y)$ is the space of 1-densities on Y. The collection of all distributions on Y with coefficients in the (finite dimensional complex vector) bundle E_0 is denoted $\mathcal{C}^{-\infty}(Y; E_0)$.

If $P = (P_x, x \in \mathcal{G}^{(0)})$ is a family of pseudodifferential operators acting on $\mathcal{G}_x$, we denote by k_x the distribution kernel of P_x

$$k_x \in \mathcal{C}^{-\infty}(\mathcal{G}_x \times \mathcal{G}_x; r_1^*(E) \otimes r_2^*(E)' \otimes \Omega_2). \tag{11}$$

Here Ω_2 is the pull-back of the bundle of vertical densities Ω_d on $\mathcal{G}_x$ to $\mathcal{G}_x \times \mathcal{G}_x$ via the second projection. These distribution kernels are obtained using Schwartz' kernel theorem. We define the support of the operator P to be

$$\operatorname{supp}(P) = \overline{\cup_x \operatorname{supp}(k_x)}. \tag{12}$$

The support of P is contained in the closed subset $\{(g, g'), d(g) = d(g')\}$ of the product $\mathcal{G}^{(1)} \times \mathcal{G}^{(1)}$. In particular, $(id \times \iota)(\operatorname{supp}(P)) \subset \mathcal{G}^{(2)}$. If all operators P_x are of order $-\infty$, then each kernel k_x is a smooth section. Actually we have more:

Lemma 1. *The collection of all distribution kernels k_x of a differentiable family $P = (P_x, x \in \mathcal{G}^{(0)})$ of order $-\infty$ operators defines a smooth section k of $r_1^*(E) \otimes r_2^*(E)' \otimes \Omega_2$ on $\{(g, g'), d(g) = d(g')\}$.*

Proof. Indeed if $\psi : d(V) \times W \to V$ is a fiber preserving diffeomorphism as in Definition 6, then it follows from the definition that k is smooth on $d(V) \times W \times W \subset \{(g, g'), d(g) = d(g')\}$. Since in this way we obtain an atlas of $\{(g, g'), d(g) = d(g')\}$, we obtain that k is smooth as claimed. □

Definition 8. The family $P = (P_x, x \in \mathcal{G}^{(0)})$ is properly supported if $p_i^{-1}(K) \cap \operatorname{supp}(P)$ is a compact set for any compact subset $K \subset \mathcal{G}$, where $p_1, p_2 : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ are the two projections. The family P is called compactly supported if its support $\operatorname{supp}(P)$ is compact; and, finally, P is called uniformly supported if its reduced support $\operatorname{supp}_\mu(P) = \mu_1(\operatorname{supp}(P))$ is a compact subset of $\mathcal{G}^{(1)}$, where $\mu_1(g', g) = g'g^{-1}$.

It immediately follows from the definition that a uniformly supported operator is also properly supported, and that a compactly supported operator is uniformly supported. If the family $P = (P_x, x \in \mathcal{G}^{(0)})$ is properly supported, then each P_x is properly supported, but the converse is not true.

Recall that the composition of two ordinary pseudodifferential operators is defined if one of them is properly supported. It follows that we can define the composition PQ of two properly supported families of operators $P = (P_x, x \in \mathcal{G}^{(0)})$ and $Q = (Q_x, x \in \mathcal{G}^{(0)})$ on $\mathcal{G}^{(1)}$ by pointwise composition $PQ = (P_x Q_x, x \in \mathcal{G}^{(0)})$. The action on sections of $r^*(E)$ is also defined pointwise as follows. For any smooth section $f \in \mathcal{C}^\infty(\mathcal{G}, r^*(E))$ denote by f_x the restriction $f|_{\mathcal{G}_x}$. If each f_x has compact support and $P = (P_x, x \in \mathcal{G}^{(0)})$ is a family of ordinary pseudodifferential operators, then we define Pf by $(Pf)_x = P_x(f_x)$.

Lemma 2. (i) *If $f \in \mathcal{C}_c^\infty(\mathcal{G}, r^*(E))$ and $P = (P_x, x \in \mathcal{G}^{(0)})$ is a differentiable family of ordinary pseudodifferential operators, then $Pf \in \mathcal{C}^\infty(\mathcal{G}, r^*(E))$. If P is also properly supported, then $Pf \in \mathcal{C}_c^\infty(\mathcal{G}, r^*(E))$.*

(ii) *The composition $PQ = (P_x Q_x, x \in \mathcal{G}^{(0)})$ of two properly supported differentiable families of operators $P = (P_x, x \in \mathcal{G}^{(0)})$ and $Q = (Q_x, x \in \mathcal{G}^{(0)})$ is a properly supported differentiable family.*

Proof. If P consists of regularizing operators, then

$$Pf(g) = \int_{\mathcal{G}_x} k_x(g,h) f(h), \quad \text{where} \quad x = d(g).$$

Lemma 1 implies that the formula above for Pf involves only the integration of smooth (uniformly in g) compactly supported sections, and hence we can exchange integration and differentiation to obtain the smoothness of Pf. This proves (i) in case P consists of regularizing operators. The proof of (ii) if both P and Q consist of regularizing operators follows the same reasoning.

We prove now (i) for P arbitrary. Fix $g \in \mathcal{G}_x$ and V a neighborhood of g fiber preserving diffeomorphic to $d(V) \times W$ for some open convex subset W in $\mathbb{R}^n$, $0 \in W$, such that $(x, 0)$ maps to g. Replacing P_x by $P_x - R_x$ for a smooth regularizing family R_x we can assume that the distribution kernels k_x of P_x satisfy

$$p_1^{-1}(d(V) \times W/4) \cap \overline{\cup \operatorname{supp}(k_x)} \subset (d(V) \times W/4) \times (d(V) \times W/2).$$

The smoothness of Pf, respectively of PQ if Q consists of regularizing operators, reduces in this way to a computation in local coordinates. This completes the proof of (i) in general, and of (ii) if Q is regularizing.

For arbitrary Q we can replace Q, in view of what has already been proved, with $Q - R$, where R is a regularizing family. In this way we may assume that

$$p_1^{-1}(d(V) \times W/2) \cap \overline{\cup \operatorname{supp}(k'_x)} \subset (d(V) \times W/2) \times (d(V) \times 3W/4),$$

where k'_x are the distribution kernels of Q_x. The support estimates above for P and Q show that the $P_y Q_y$ for $y \in d(V)$ are the compositions of smooth families of pseudodifferential operators acting on $W \subset \mathbb{R}^n$. The result is then known. □

The smaller class of uniformly supported operators is also closed under composition.

Lemma 3. *The composition $PQ = (P_x Q_x, x \in \mathcal{G}^{(0)})$ of two uniformly supported families of operators $P = (P_x, x \in \mathcal{G}^{(0)})$ and $Q = (Q_x, x \in \mathcal{G}^{(0)})$ is uniformly supported.*

Proof. The reduced support $\operatorname{supp}_\mu(PQ)$ (see (12)) of the composition PQ satisfies

$$\operatorname{supp}_\mu(PQ) \subset \mu\big(\operatorname{supp}_\mu(P) \times \operatorname{supp}_\mu(Q)\big),$$

where μ is the composition of arrows. Since $\operatorname{supp}_\mu(P)$ and $\operatorname{supp}_\mu(Q)$ are compact, the equation above completes the proof of the lemma. □

Let $\mathcal{G}$ be an almost differentiable groupoid. The space of order m, invariant, *uniformly* supported pseudodifferential operators on $\mathcal{G}$, acting on sections of the vector bundle E will be denoted by $\Psi^m(\mathcal{G}; E)$. We denote $\Psi^\infty(\mathcal{G}; E) = \cup_{m\in\mathbb{Z}} \Psi^m(\mathcal{G}; E)$ and $\Psi^{-\infty}(\mathcal{G}; E) = \cap_{m\in\mathbb{Z}} \Psi^m(\mathcal{G}; E)$. Thus an operator $P \in \Psi^m(\mathcal{G}; E)$ is actually a differentiable family $P = (P_x, x \in \mathcal{G}^{(0)})$ of ordinary pseudodifferential operators.

Theorem 1. *The set $\Psi^\infty(\mathcal{G}; E)$ of uniformly supported invariant pseudodifferential operators on an almost differentiable groupoid $\mathcal{G}$ is a filtered algebra, i.e.*

$$\Psi^m(\mathcal{G}; E)\Psi^{m'}(\mathcal{G}; E) \subset \Psi^{m+m'}(\mathcal{G}; E).$$

In particular, $\Psi^{-\infty}(\mathcal{G}; E)$ is a two-sided ideal.

Proof. Let $P = (P_x, x \in \mathcal{G}^{(0)})$ and $Q = (Q_x, x \in \mathcal{G}^{(0)})$ be two invariant uniformly supported pseudodifferential operators on $\mathcal{G}$, of order m and m' respectively. Their composition $PQ = (P_x Q_x)$, is a uniformly supported operator of order $m + m'$, in view of Lemma 3. It is also a differentiable family due to Lemma 2. We now check the invariance condition. Let g be an arbitrary arrow and $U_g : \mathcal{C}_c^\infty(\mathcal{G}_x, r^*(E)) \to \mathcal{C}_c^\infty(\mathcal{G}_y, r^*(E))$, $x = d(g)$ and $y = r(g)$, be as in the definition above. Then

$$(PQ)_y U_g = P_y Q_y U_g = P_y U_g Q_x = U_g P_x Q_x = U_g (PQ)_x.$$

This proves the theorem. □

Properly supported invariant differentiable families of pseudodifferential operators also form a filtered algebra, denoted $\Psi^\infty_{\mathrm{prop}}(\mathcal{G}; E)$. While it is clear that in order for our class of pseudodifferential operators to form an algebra we need some condition on the support of their distribution kernels, exactly what support condition to impose is a matter of choice. We prefer the uniform support condition because it leads to a better control at infinity of the family of operators $P = (P_x, x \in \mathcal{G}^{(0)})$ and allows us to identify the regularizing ideal (i.e. the ideal of order $-\infty$ operators) with the groupoid convolution algebra of $\mathcal{G}$. The choice of uniform support will also ensure that $\Psi^m(\mathcal{G}; E)$ behaves functorially with respect to open embeddings. The compact support condition enjoys the same properties but is usually too restrictive. The issue of support will be discussed again in examples.

The definition of the principal symbol extends easily to $\Psi^m(\mathcal{G}; E)$. Denote by $\pi : A^*(\mathcal{G}) \to M$, $(M = \mathcal{G}^{(0)})$ the projection. If $P = (P_x, x \in \mathcal{G}^{(0)}) \in \Psi^m(\mathcal{G}; E)$ is an order m pseudodifferential differential operator on $\mathcal{G}$, then the principal symbol $\sigma_m(P)$ of P will be represented by sections of the bundle $\mathrm{End}(\pi^* E)$ and will be defined to satisfy

$$\sigma_m(P)(\xi) = \sigma_m(P_x)(\xi) \in \mathrm{End}(E_x) \quad \text{if} \quad \xi \in A^*_x(\mathcal{G}) = T^*_x \mathcal{G}_x \tag{13}$$

(the equation above is $\mod \mathcal{S}^{m-1}_{\mathrm{cl}}(A^*_x(\mathcal{G}); \mathrm{End}(E)))$. This equation will obviously uniquely determine a linear map

$$\sigma_m : \Psi^m(\mathcal{G}) \to \mathcal{S}^m_{\mathrm{cl}}(A^*(\mathcal{G}); \mathrm{End}(E))/\mathcal{S}^{m-1}_{\mathrm{cl}}(A^*(\mathcal{G}); \mathrm{End}(E))$$

provided we can show that for any $P = (P_x, x \in \mathcal{G}^{(0)})$ there exists a symbol $a \in \mathcal{S}^m_{\mathrm{cl}}(A^*(\mathcal{G}); \mathrm{End}(E))$ whose restriction to $A^*_x(\mathcal{G})$ is a representative of the principal symbol of P_x in that fiber for each x. We thus need to choose for each P_x a representative $a_x \in \mathcal{S}^m_{\mathrm{cl}}(A^*_x(\mathcal{G}); \mathrm{End}(E))$ of $\sigma_m(P_x)$ such that the family a_x is smooth and invariant. Assume first that E is the trivial line bundle and proceed as in [**14**] Section 18.1, especially Equation (18.1.27) and below.

Choose a connection ∇ on the vector bundle $A(\mathcal{G}) \to \mathcal{G}^{(0)}$ and consider the pull-back vector bundle $r^*(A) \to \mathcal{G}$ of $A(\mathcal{G}) \to \mathcal{G}^{(0)}$ endowed with the pull-back connection $\widetilde{\nabla} = r^*\nabla$. Its restriction on any fiber $\mathcal{G}_x$ defines a linear connection in the usual sense, which is denoted by ∇_x. These connections are right invariant in the sense that

$$R^*_g \nabla_x = \nabla_y, \quad \forall g \in \mathcal{G} \text{ such that } r(g) = x \text{ and } d(g) = y. \tag{14}$$

Using such an invariant connection, we may define the exponential map of a Lie algebroid, which generalizes the usual exponential map of a manifold with a connection and the exponential map of a Lie algebra as follows. For any $x \in \mathcal{G}^{(0)}$, define a map $\exp_x : A_x \to \mathcal{G}$ as the composition of the maps:

$$A_x \xrightarrow{i} T_x\mathcal{G}_x \xrightarrow{\tilde{\exp}_x} \mathcal{G},$$

where i is the natural inclusion and $\tilde{\exp}_x = \exp_{\nabla_x}$ is the usual exponential map at $x \in \mathcal{G}_x$ on the manifold $\mathcal{G}_x$. By varying the point x, we obtain a map

$\exp_\nabla$ defined in a neighborhood of the zero section, called the *exponential map* of the Lie algebroid[2] . Clearly, $\exp_\nabla$ is a local diffeomorphism

$$A(\mathcal{G}) \supset V_0 \ni v \longrightarrow \exp_\nabla(v) = y \in V \subset \mathcal{G} \tag{15}$$

mapping an open neighborhood V_0 of the zero section in $A(\mathcal{G})$ diffeomorphically to a neighborhood V of $\mathcal{G}^{(0)}$ in $\mathcal{G}$, and sending the zero section onto the set of units. Choose a cut-off function $\phi \in \mathcal{C}^\infty(\mathcal{G})$ with support in V and equal to 1 in a smaller neighborhood of $\mathcal{G}^{(0)}$ in $\mathcal{G}$. If $y \in V$, $x = d(y)$ and $\xi \in A_x^*(\mathcal{G})$ let $v \in V_0$ be the unique vector $v \in A_x(\mathcal{G})$ such that $y = \exp_\nabla(v)$ and denote $e_\xi(y) = \phi(y)e^{iv\cdot\xi}$, which extends then to all $y \in \mathcal{G}$ due to the cut-off function ϕ. Define the (∇, ϕ)–complete symbol $\sigma_{\nabla,\phi}(P)$ by

$$\sigma_{\nabla,\phi}(P)(\xi) = (P_x e_\xi)(x), \qquad \forall \xi \in T_x^*\mathcal{G}_x = A_x^*(\mathcal{G}). \tag{16}$$

Lemma 4. *If $P = (P_x, x \in \mathcal{G}^{(0)})$ is an operator in $\Psi^m(\mathcal{G})$, then the function $\sigma_{\nabla,\phi}(P)$ defined above is differentiable and defines a symbol in $\mathcal{S}^m_{\mathrm{cl}}(A^*(\mathcal{G}))$. Moreover, if (∇_1, ϕ_1) is another pair consisting of an invariant connection ∇_1 and a cut-off function ϕ_1, then $\sigma_{\nabla,\phi}(P) - \sigma_{\nabla,\phi_1}(P)$ is in $\mathcal{S}^{-\infty}_{\mathrm{cl}}(A^*(\mathcal{G}))$ and $\sigma_{\nabla,\phi}(P) - \sigma_{\nabla_1,\phi_1}(P)$ is in $\mathcal{S}^{m-1}_{\mathrm{cl}}(A^*(\mathcal{G}))$.*

Proof. For each $\xi \in A_x^*$ the function e_ξ is smooth with compact support on $\mathcal{G}_x$ so $P_x e_\xi$ is defined. Equation (18.1.27) of [**14**] shows that $a(\xi) = \sigma_{\nabla,\phi}(P)(\xi)$ is the restriction of the complete symbol of $P_x\phi$ to $T_x^*\mathcal{G}_x$ if the complete symbol is defined in the normal coordinate system at $x \in \mathcal{G}_x$ (given by the exponential map). The normal coordinate system defines, using a local trivialization of $A(\mathcal{G})$, a fiber preserving diffeomorphism $\psi : d(V)\times W \to V$ for some open subset W of $\mathbb{R}^n$ (i.e. satisfying $d(\psi(x,w)) = x$). From the definition of the smoothness of the family P_x (Definition 6) it follows that the complete symbol of $P\phi$ is in $\mathcal{S}^m_{\mathrm{cl}}(d(V)\times T^*W)$ if the support of ϕ is chosen to be in V. This proves that $\sigma_{\nabla,\phi}(P)$ is in $\mathcal{S}^m_{\mathrm{cl}}(A^*(\mathcal{G}))$.

The rest follows in exactly the same way. □

The lemma above justifies the following definition of the principal symbol as the class of $\sigma_{\nabla,\phi}(P)$ modulo terms of lower order (for the trivial line bundle $E = \mathbb{C}$). This definition will be, in view of the same lemma, independent on the choice of ∇ or ϕ and will satisfy Equation (13). If E is not trivial one can still define a complete symbol $\sigma_{\nabla,\nabla',\phi}(P)$, depending also on a second connection ∇' on the bundle E, which is used to trivialize $r^*(E)$ on $V \subset \mathcal{G}$ (assuming also that V_0 is convex). Alternatively, we can use Proposition 3 below.

[2]See [**17**] for an alternative definition of the exponential map. One should not confuse this map with the exponential map from $\Gamma(A)$ to the bisections of the groupoid as defined in [**16**].

Proposition 1. *Let ∇ and ϕ be as above. The choice of a connection ∇' on E defines a complete symbol map $\sigma_{\nabla,\nabla',\phi} : \Psi^m(\mathcal{G}; E) \to \mathcal{S}^m_{\rm cl}(A^*(\mathcal{G}))$. The principal symbol $\sigma_m : \Psi^m(\mathcal{G}; E) \to \mathcal{S}^m_{\rm cl}(A^*(\mathcal{G}))/\mathcal{S}^{m-1}_{\rm cl}(A^*(\mathcal{G}))$, defined by*

$$\sigma_m(P) = \sigma_{\nabla,\nabla',\phi}(P) + \mathcal{S}^{m-1}_{\rm cl}(A^*(\mathcal{G})), \tag{17}$$

does not depend on the choice of the connections ∇, ∇' or the cut-off function ϕ.

Proof. The (∇, ∇', ϕ)–complete symbol $\sigma_{\nabla,\nabla',\phi}(P)$ is defined as follows. Let w be a vector in E_x. Using the connection ∇' we can define a section $\tilde{w}$ of $r^*(E)$ on $\mathcal{G}_x \cap V$ by parallel transport along the geodesics of ∇ starting at x, and which coincides with w at x. Then denote $e_{\xi,w} = e_\xi \tilde{w}$ and let

$$\sigma_{\nabla,\nabla',\phi}(P)(\xi)w = (P_x e_{\xi,w})(x) \in E_x, \qquad \forall \xi \in T^*_x\mathcal{G}_x = A^*_x(\mathcal{G}). \tag{18}$$

The rest of the proof proceeds along the lines of the proof of Lemma 4. □

Note that the principal symbol of P determines the principal symbols of the individual operators P_x by the invariance with respect to right translations. Precisely, we have $\sigma_m(P_x) = r^*(\sigma(P))|_{T^*\mathcal{G}_x}$.

The following result extends some very well known properties of the calculus of pseudodifferential operators on smooth manifolds. We shall prove the surjectivity of the principal symbol in Section 5.

Proposition 2. (i) *The principal symbol map*

$$\sigma_m : \Psi^m(\mathcal{G}; E) \to \mathcal{S}^m_{\rm cl}(\mathcal{G}; \mathrm{End}(E))/\mathcal{S}^{m-1}_{\rm cl}(\mathcal{G}; \mathrm{End}(E))$$

has kernel $\Psi^{m-1}(\mathcal{G}; E)$ and satisfies Equation (13).

(ii) *The composition PQ of two operators $P, Q \in \Psi^\infty(\mathcal{G}; E)$, of orders m and, respectively, m', satisfies $\sigma_{m+m'}(PQ) = \sigma_m(P)\sigma_{m'}(Q)$.*

Proof. (i) The operator $P = (P_x, x \in \mathcal{G}^{(0)}) \in \Psi^m(\mathcal{G}; E)$ is in the kernel of σ_m if and only if all symbols $\sigma_m(P_x)$ vanish. This implies $P_x \in \Psi^{m-1}(\mathcal{G}; E)$ for all x and hence $P = (P_x, x \in \mathcal{G}^{(0)}) \in \Psi^{m-1}(\mathcal{G}; E)$. As already observed for E a trivial line bundle, the fact that Equation (13) is satisfied was contained in the proof of Lemma 4. The general case is similar or can be proved using Proposition 3.

The second statement is known for pseudodifferential operators on smooth manifolds [**14**]; this accounts for the second equality sign in the next equation. We obtain using Equation (13) that

$$\begin{aligned}\sigma_{m+m'}(PQ)(v) = \sigma_{m+m'}(P_xQ_x)(v) &= \sigma_m(P_x)\sigma_{m'}(Q_x)(v)\\ &= \sigma_m(P_x)(v)\sigma_{m'}(Q_x)(v),\end{aligned}$$

where $v \in A^*_x(\mathcal{G})$. □

Although for the most of this paper we shall be concerned with groupoids, the definition of $\Psi^m(\mathcal{G}; E)$ easily extends to *local* groupoids. Indeed it suffices to modify the invariance condition in Definition 7, using the notation in Equation (7), as follows. We assume that for any $g \in \mathcal{G}^{(1)}$ and any smooth compactly supported function ϕ on $\mathcal{U}_g$ there exists a regularizing operator $R_{g,\phi}$ such that

$$U_g(\phi)P_{r(g)}U_g f - U_g(\phi P_{d(g)} f) = R_{g,\phi} f \tag{19}$$

for any function $f \in \mathcal{C}^\infty_c(\mathcal{U}_g)$. We thus replace the strict invariance of the original definition by 'invariance up to regularizing operators'.

We denote by $\Psi^m_{\mathrm{loc}}(\mathcal{G}; E)$ the set of differentiable properly supported families $P = (P_x, x \in \mathcal{G}^{(0)})$ of order m pseudodifferential operators satisfying the condition (19) above. Note that, if we regard an almost differentiable groupoid $\mathcal{G}$ as a local groupoid, then $\Psi^\infty(\mathcal{G}; E) \subset \Psi^m_{\mathrm{loc}}(\mathcal{G}; E)$. The inclusion is generally a strict one, though, because Equation (19) gives no condition for order $-\infty$ operators, and so $\Psi^{-\infty}_{\mathrm{loc}}(\mathcal{G}; E)$ consists of arbitrary smooth families $P = (P_x, x \in \mathcal{G}^{(0)})$ of regularizing operators. This ideal is too big to reflect the structure of $\mathcal{G}$. The "symbolic" part remains however the same:

$$\Psi^m_{\mathrm{prop}}(\mathcal{G}; E)/\Psi^{-\infty}_{\mathrm{prop}}(\mathcal{G}; E) \simeq \Psi^m_{\mathrm{loc}}(\mathcal{G}; E)/\Psi^{-\infty}_{\mathrm{loc}}(\mathcal{G}; E).$$

If the sets $\mathcal{U}_g$ are all connected (in which case the local groupoid $\mathcal{G}$ is said to be d–connected) an easier condition to use than (19) is

$$[X, P] \in \Psi^{-\infty}_{\mathrm{loc}}(\mathcal{G}; E) \tag{20}$$

for all r–vertical left–invariant vector fields X on $\mathcal{G}^{(1)}$. With this, the following analog of Theorem 1, becomes straightforward.

Theorem 2. *Assume that $\mathcal{G}$ is a d–connected almost differentiable groupoid. Then the space $\Psi^\infty_{\mathrm{loc}}(\mathcal{G})$ is a filtered algebra, with $\Psi^{-\infty}(\mathcal{G}; E)$ as residual ideal.*

Proof. The only thing to check is that $\Psi^\infty_{\mathrm{loc}}(\mathcal{G}; E)$ is closed under composition. The composition of two differentiable, properly supported families $P, Q \in \Psi^\infty_{\mathrm{loc}}(\mathcal{G}; E)$ is again differentiable and properly supported, as has already been proved. The infinitesimal invariance condition $[X, PQ] = [X, P]Q + P[X, Q] \in \Psi^{-\infty}_{\mathrm{loc}}(\mathcal{G}; E)$ (20) follows from the fact that $\Psi^{-\infty}_{\mathrm{loc}}(\mathcal{G}; E)$ is an ideal of $\Psi^\infty_{\mathrm{loc}}(\mathcal{G}; E)$. □

3. Differential operators and quantization.

In this section, we examine the differential operators in $\Psi^\infty(\mathcal{G}; E)$, if $\mathcal{G}$ is a global groupoid, or in $\Psi^\infty_{\mathrm{loc}}(\mathcal{G}; E)$ if $\mathcal{G}$ is a local groupoid. We also show how a simple algebraic construction applied to $\mathcal{G}$ and to the algebras $\Psi^\infty_{\mathrm{loc}}(\mathcal{G}; E)$ leads to a concrete construction of a deformation quantization of the Lie-Poisson structure on the dual of a Lie algebroid.

In this section, $\mathcal{G}$ will be an almost differentiable local groupoid. This generality is necessary in order to integrate arbitrary Lie algebroids. Nevertheless, when A is the Lie algebroid of an almost differentiable *global* groupoid $\mathcal{G}$ (that is not just a *local* groupoid), then all results we shall prove for $\Psi^\infty_{\mathrm{loc}}(\mathcal{G};E)$ in this section extend immediately to $\Psi^\infty(\mathcal{G};E)$, although we shall not mention this each time.

Lemma 5. *Let $P = (P_x, x \in \mathcal{G}^{(0)})$ be an operator in $\Psi^\infty_{\mathrm{loc}}(\mathcal{G};E)$. If P_x is a multiplication operator for all x, then there exists a smooth endomorphism s of E such that $P_x(g) = s(r(g))$ for all $g \in \mathcal{G}_x$. Conversely, every smooth section s of $\mathrm{End}(E)$ defines a multiplication operator in $\Psi^0_{\mathrm{loc}}(\mathcal{G};E)$.*

Proof. By assumption $P_x(g)$ is in $\mathrm{End}(E_{r(g)})$. The invariance relation shows that $P_x(g)$ depends only on $r(g)$. This defines the section s of $\mathrm{End}(E)$ such that $P_x(g) = s(r(g))$. To show that s is smooth, we let ϕ be a smooth section of E over $\mathcal{G}^{(0)}$ and let $\tilde{\phi}(g) = \phi(r(g))$. By assumption $P\tilde{\phi}$ is smooth and hence $s\phi = P\tilde{\phi}|_{\mathcal{G}^{(0)}}$ is also smooth. Since ϕ is arbitrary this implies the smoothness of s.

Conversely, if s is a smooth endomorphism of E, then if we let $P_x(g) = s(r(x))$ we obtain a multiplication operator in $\Psi^0_{\mathrm{loc}}(\mathcal{G};E)$. □

The following proposition will allow us to assume that E is a trivial bundle, which is sometimes useful in applications.

Proposition 3. *Let E be a vector bundle on $\mathcal{G}^{(0)}$ embedded into a trivial hermitian bundle, $E \subset \mathbb{C}^N$. Denote by e_0 the projection onto E regarded as a matrix of multiplication operators in $M_N(\Psi^0_{\mathrm{loc}}(\mathcal{G}))$, the algebra of $N \times N$ matrices with values in $\Psi^0_{\mathrm{loc}}(\mathcal{G})$. Then $\Psi^\infty_{\mathrm{loc}}(\mathcal{G};E) \simeq e_0 M_N(\Psi^\infty_{\mathrm{loc}}(\mathcal{G}))e_0$ as filtered algebras.*

Proof. The multiplication operator e_0 defines an element of $\Psi^0_{\mathrm{loc}}(\mathcal{G};E)$ by the lemma above; hence it acts on all spaces $\mathcal{C}^\infty_{\mathrm{c}}(\mathcal{G}_x, \mathbb{C}^N)$. Then

$$\mathcal{C}^\infty_{\mathrm{c}}(\mathcal{G}_x, r^*(E)) = e_0 \mathcal{C}^\infty_{\mathrm{c}}(\mathcal{G}_x, \mathbb{C}^N)$$

and every pseudodifferential operator P_x on $\mathcal{C}^\infty_{\mathrm{c}}(\mathcal{G}_x, r^*(E))$ extends in this way to an operator on $\mathcal{C}^\infty_{\mathrm{c}}(\mathcal{G}_x, \mathbb{C}^N)$. This gives an inclusion $\Psi^\infty_{\mathrm{loc}}(\mathcal{G};E) \subset e_0 M_N(\Psi^\infty_{\mathrm{loc}}(\mathcal{G}))e_0$. Conversely if P_x is a pseudodifferential operator on $\mathcal{C}^\infty_{\mathrm{c}}(\mathcal{G}_x, \mathbb{C}^N)$, then $e_0 P_x e_0$ is a pseudodifferential operator on $\mathcal{C}^\infty_{\mathrm{c}}(\mathcal{G}_x, \mathbb{C}^N)$. This gives the opposite inclusion. □

The following proposition shows the intimate connection between $A(\mathcal{G})$, the Lie algebroid of $\mathcal{G}$, and $\Psi^\infty_{\mathrm{loc}}(\mathcal{G})$ ($\Psi^\infty(\mathcal{G})$ if $\mathcal{G}$ is global). It is morally an equivalent definition of the Lie algebroid associated to an almost differentiable local groupoid.

Proposition 4. *Let $\mathcal{G}$ be an almost differentiable local groupoid.*

(i) *The algebra* $\mathcal{C}^\infty(\mathcal{G}^{(0)})$ *is the algebra of multiplication operators in* $\Psi^0_{\mathrm{loc}}(\mathcal{G})$.
(ii) *The space of sections of the Lie algebroid* $A(\mathcal{G})$ *can be identified with the space of order* 1 *differential operators in* $\Psi^1_{\mathrm{loc}}(\mathcal{G})$ *without constant term.*
(iii) *The Lie algebroid structure of* $A(\mathcal{G})$ *is induced by the commutator operations* $[\ ,\ ] : \Psi^1_{\mathrm{loc}}(\mathcal{G}) \times \Psi^1_{\mathrm{loc}}(\mathcal{G}) \to \Psi^1_{\mathrm{loc}}(\mathcal{G})$ *and* $[\ ,\ ] : \Psi^1_{\mathrm{loc}}(\mathcal{G}) \times \Psi^0_{\mathrm{loc}}(\mathcal{G}) \to \Psi^0_{\mathrm{loc}}(\mathcal{G})$.

Proof. The first part is a particular case of Lemma 5, only easier. Order 1 differential operators without constant term are vector fields, right invariant by the definition of $\Psi^1_{\mathrm{loc}}(\mathcal{G})$, so they can be identified with the sections of the Lie algebroid $A(\mathcal{G})$ of $\mathcal{G}$. This proves (ii). In order to check (iii) recall that, if we regard vector fields on $\mathcal{G}$ as linear maps $\mathcal{C}^\infty(\mathcal{G}) \to \mathcal{C}^\infty(\mathcal{G})$, then the Lie bracket coincides with the commutator of linear maps. Moreover the commutator $[X, f]$ of a vector field X and of a multiplication map f is $[X, f] = X(f)$, again regarded as a linear map. Then (iii) follows in view of the discussion above. □

The Lie algebroid $A = A(\mathcal{G})$ turns out to determine the structure of the algebra of invariant tangential differential operators on $\mathcal{G}$, denoted $\mathrm{Diff}(\mathcal{G})$. We shall see that the subalgebra $\mathrm{Diff}(\mathcal{G}) \subset \Psi^\infty_{\mathrm{loc}}(\mathcal{G})$ is a concrete model of the *universal enveloping algebra* of the Lie algebroid A [**15**, **36**], a concept whose definition we now recall.

Given a Lie algebroid $A \to M$ with anchor ρ, we can make the $C^\infty(M)$-module direct sum $C^\infty(M) \oplus \Gamma(A)$ into a Lie algebra over $\mathbb{C}$ by defining

$$[f + X,\ g + Y] = (\rho(X)g - \rho(Y)f) + [X, Y].$$

Let $U = U(C^\infty(M) \oplus \Gamma(A))$ be its universal enveloping algebra. For any $f \in C^\infty(M)$ and $X \in \Gamma(A)$, denote by f' and X' their canonical image in U. Denote by I the two-sided ideal of U generated by all elements of the form $(fg)' - f'g'$ and $(fX)' - f'X'$. Define

$$U(A) = U/I. \tag{21}$$

$U(A)$ is called the universal enveloping algebra of the Lie algebroid A. When A is a Lie algebra, this definition reduces to the usual universal enveloping algebra. We shall see, for example, that for the tangent bundle TM this is the algebra of differential operators on M.

The maps $f \to f'$ and $X \to X'$ considered above descend to linear embeddings $i_1 : C^\infty(M) \to U(A)$, and $i_2 : \Gamma(A) \to U(A)$; the first map i_1 is an algebra morphism. These maps have the following properties:

$$\begin{gathered} i_1(f)i_2(X) = i_2(fX), \quad [i_2(X), i_1(f)] = i_1(\rho(X)f), \\ [i_2(X), i_2(Y)] = i_2([X, Y]). \end{gathered} \tag{22}$$

In fact, $U(A)$ is universal among triples (B, ϕ_1, ϕ_2) having these properties (see [15] for a proof of this easy fact).

In particular, if M is the space of units of an almost differentiable groupoid $\mathcal{G}$ with Lie algebroid $A(\mathcal{G})$, the natural morphisms $\phi_1 : C^\infty(M) \to \mathrm{Diff}(\mathcal{G})$ and $\phi_2 : \Gamma(A) \to \mathrm{Diff}(\mathcal{G})$ obtained from Proposition 4 extend to a unique algebra morphism $\tau : U(A) \to \mathrm{Diff}(\mathcal{G})$. (Recall that we denoted by $\mathrm{Diff}(\mathcal{G})$ the algebra of right invariant tangential differential operators on $\mathcal{G}$.) Denote by $U_n(A) \subset U(A)$ the space generated by $\mathcal{C}^\infty(M)$ and the images of $X_1 \otimes X_2 \otimes \ldots \otimes X_k \in U = U(C^\infty(M) \oplus \Gamma(A))$, for $k \leq n$, under the canonical projection $U \to U(A) = U/I$. Then

$$U_0(A) \subset U_1(A) \subset \cdots \subset U_n(A) \subset \cdots \tag{23}$$

is a filtration of $U(A)$. The relations (22) show that, as in the Lie algebra case, the graded algebra $\oplus U_n(A)/U_{n-1}(A)$ is commutative. Similarly, $\mathrm{Diff}(\mathcal{G})$ is naturally filtered by degree.

Lemma 6. *The map $\tau : U(A) \to \mathrm{Diff}(\mathcal{G})$ maps $U_n(A)$ onto the space $\mathrm{Diff}_n(\mathcal{G})$ of operators of order $\leq n$.*

Proof. Let $D \in \mathrm{Diff}(\mathcal{G})$ be an invariant tangetial differential operator of order $\leq n$. By right invariance D is completely determined by the restrictions $(Du)|_{\mathcal{G}^{(0)}}$, $u \in \mathcal{C}_c^\infty(\mathcal{G})$. Since D acts on the fibers of d we can write

$$(Du)|_{\mathcal{G}^{(0)}} = \sum_{i=1}^{n} D_i u,$$

where D_i is a superposition of derivations $D_i u = X_1^{(i)} X_2^{(i)} \cdots X_{k_i}^{(i)} u$ defined using the tangential derivations $X_j^{(i)} \in \Gamma(A)$. By definition it follows that D is the sum of $\tau\left(X_1^{(i)} X_2^{(i)} \cdots X_{k_i}^{(i)}\right)$. □

Denote by $\mathrm{Symm}(A)$ the *symmetric* tensor product of the bundle A, that is

$$\mathrm{Symm}(A) = \bigoplus_{n=0}^{\infty} S_n(A),$$

where $S_n(A)$ is the symmetric quotient of the bundle $A^{\otimes n}$, and is isomorphic to the subspace of symmetric tensors, if $S_0(A)$ is the trivial $\mathbb{R}$ bundle by convention. The space $\Gamma(\mathrm{Symm}(A))$ of smooth sections of $\mathrm{Symm}(A)$ identifies with the space of smooth functions on A^* polynomial in each fiber. The complete symbol map $\sigma_{\nabla,\phi}(D)$ of an invariant differential operator $\mathrm{Diff}(\mathcal{G}) \subset \Psi^\infty_{\mathrm{loc}}(\mathcal{G})$ (defined in Equation (16)) does not depend on the cut-off function ϕ and will be a polynomial in ξ, denoted simply by $\sigma_\nabla(D)$.

Using the algebra morphism $\tau : U(A) \to \mathrm{Diff}(\mathcal{G})$ obtained from the universality property of $U(A)$, we have the following Poincaré-Birkhoff-Witt type theorem for Lie algebroids. Note that both $U(A)$ and $\Gamma(\mathrm{Symm}(A))$ have natural filtrations (see (23)).

Theorem 3 (Poincaré-Birkhoff-Witt). *The composite map,*

$$U(A) \ni D \to \sigma_\nabla(\tau(D)) \in \Gamma(\mathrm{Symm}(A)),$$

is an isomorphism of filtered vector spaces. In particular, $\tau : U(A) \to \mathrm{Diff}(\mathcal{G})$ is an algebra isomorphism.

Proof. It follows from definitions that the map $\sigma = \sigma_\nabla \circ \tau$ considered in the statement maps $U_n(A)$ to $\oplus_{k=0}^n \Gamma(S_k(A))$ and hence it preserves the filtration. By abuse of notation we shall still denote by σ the induced map $U_n(A)/U_{n-1}(A) \to \Gamma(S_n(A))$. It is enough to prove that the map of graded spaces

$$\sigma : \bigoplus U_n(A)/U_{n-1}(A) \to \bigoplus \Gamma(S_n(A)) = \Gamma(\mathrm{Symm}(A))$$

is an isomorphism. By Lemma 6 this map is onto. We now prove that it is one-to-one.

The inclusion of $\mathcal{C}^\infty(M)$ in $U(A)$ makes $U(A)$ a $\mathcal{C}^\infty(M)$-bimodule. The filtration $U_n(A)$ of $U(A)$ consists of $\mathcal{C}^\infty(M)$-bimodules. Moreover, since the graded algebra $\oplus U_n(A)/U_{n-1}(A)$ is commutative, the quotient $U_n(A)/U_{n-1}(A)$ consists of central elements for this action (i.e. the left and right $\mathcal{C}^\infty(M)$-module structure coincide). It follows from the definition that the subspace $\Gamma(A)^{\otimes n}$ of the universal enveloping algebra $U = U(C^\infty(M) \oplus \Gamma(A))$ maps onto $U_n(A)/U_{n-1}(A)$. The previous discussion shows that this map descends to a map from the tensor product $\Gamma(A) \otimes_{\mathcal{C}^\infty(M)} \cdots \otimes_{\mathcal{C}^\infty(M)} \Gamma(A)$ of $\mathcal{C}^\infty(M)$-modules. By the commutativity of the graded algebra of $U(A)$ this further descends to a $\mathcal{C}^\infty(M)$-linear surjective map $q : \Gamma(S_n(A)) \to U_n(A)/U_{n-1}(A)$.

The composition $\sigma \circ q : \Gamma(\mathrm{Symm}(A)) \longrightarrow \Gamma(\mathrm{Symm}(A))$ is multiplicative since both q and σ are multiplicative. Moreover $\sigma \circ q$ is the identity when restricted to $\mathcal{C}^\infty(M)$ (the order 0 elements) and $\Gamma(A)$ (the elements of order 1). Since these form a system of generators of the commutative algebra $\Gamma(\mathrm{Symm}(A))$ it follows that $\sigma \circ q$ is the identity. This completes the proof. □

Remark. The Poincaré-Birkhoff-Witt theorem was proved in the algebraic context by Rinehart [**36**] for (L, R)-algebras (an algebraic version of Lie algebroids). It essentially stated that the associated graded algebra $grU(A) = \oplus_n U_{n+1}(A)/U_n(A)$ is isomorphic to the symmetric algebra $S(\Gamma(A)) = \Gamma(\mathrm{Symm}(A))$. The role of the connection ∇ on $A \to M$ is to establish an *explicit* isomorphism $\sigma_\nabla \circ \tau$ between $U(A)$ and $\Gamma(\mathrm{Symm}(A))$.

We will now use the results of this and the previous section to obtain an explicit deformation quantization of A^*. In order to do that we need to establish the relation between commutators and the Poisson bracket in our calculus.

For any $x \in \mathcal{G}^{(0)}$, $T^*\mathcal{G}_x$ is a symplectic manifold, so $T^*_d\mathcal{G} \stackrel{\text{def}}{=} \cup_{x\in\mathcal{G}^{(0)}} T^*\mathcal{G}_x$ is a regular Poisson manifold with the leafwise symplectic structures. Now the Poisson structure on A^* can be considered as being induced from that on $T^*_d\mathcal{G}$. More precisely, let $\Phi : T^*_d\mathcal{G} \to A^*$ be the natural projection induced by the right translation, used to define a map $\Phi^* : \mathcal{C}^\infty(A^*(\mathcal{G})) \to \mathcal{C}^\infty(T^*_d\mathcal{G})$. We then have:

Lemma 7. *The map Φ is a Poisson map.*

Of course this lemma is really the definition of the Poisson structure on A^*. The point is to show that the subspace $\Phi^*(\mathcal{C}^\infty(A^*(\mathcal{G})))$ of $\mathcal{C}^\infty(T^*_d\mathcal{G})$ is closed under the Poisson bracket.

Proof. It is enough to check that

$$\Phi^*(\{f, g\}) = \{\Phi^*(f), \Phi^*(g)\}, \tag{24}$$

where f and g are two smooth function on A^* with *polynomial* restrictions on each fiber of A^*, that is for f and g in $\Gamma(\mathrm{Symm}(A))$. Since the Poisson bracket is a derivation in each variable it is further enough to check this for f constant or f linear in each fiber. If both f and g are constant in each fiber, then both sides of Equation (24) vanish. If f and g are of degree one in each fiber, then they correspond to sections X and Y of A, and their Poisson bracket will identify to $[X, Y]$ (so in particular will also be of degree one in each fiber and this justifies the name of Lie-Poisson structure for this Poisson structure). For this situation the relation (24) follows from the identification of $\Gamma(A)$ with d-vertical right invariant vector fields on $\mathcal{G}$ and the fact that $T^*_d\mathcal{G}$ is a Lie-Poisson manifold itself. The remaining case is treated similarly. □

We shall use the following general fact about the principal symbols of commutators.

Proposition 5. *When E is the trivial line bundle, the commutator $[P, Q]$ satisfies $\sigma_{m+m'-1}([P, Q]) = \{\sigma_m(P), \sigma_{m'}(Q)\}$, where $\{\ ,\ \}$ is the Poisson structure on $A^*(\mathcal{G})$.*

Proof. The map $\Phi^* : \mathcal{C}^\infty(A^*(\mathcal{G})) \to \mathcal{C}^\infty(T^*_d\mathcal{G})$ is a Poisson map according to Lemma 7. Since $\Phi^*(\sigma_m(P)) = \sigma_m(P_x)$ on $T^*\mathcal{G}_x$ the result follows from

$$\begin{aligned}\Phi^*(\sigma_{m+m'-1}([P, Q])) &= \sigma_{m+m'-1}([P_x, Q_x]) = \{\sigma_m(P_x), \sigma_{m'}(Q_x)\}\\ &= \{\Phi^*(\sigma_m(P)), \Phi^*(\sigma_{m'}(Q))\}\\ &= \Phi^*(\{\sigma_m(P), \sigma_{m'}(Q)\}).\end{aligned}$$

Since Φ^* is one-to-one this proves the last statement. □

We now use the results above to construct deformation quantizations.

Deform the Lie bracket structure on the Lie algebroid A on M to obtain a new algebroid, the *adiabatic* algebroid A_t associated to A, defined over $M\times[0,\infty)$ as follows. As a bundle A_t is the lift of the bundle A to $M\times[0,\infty)$. Regard the sections X of A_t as functions $X:[0,\infty)\to\Gamma(A)$, $t\to X_t$. Then the algebroid structure is obtained by letting

$$\begin{aligned} [X,Y]_t &= t[X_t,Y_t] \qquad \text{and} \\ \rho(X)_t &= t\rho(X_t) \end{aligned}$$

so that $\rho(X)f$ is the function whose restriction to $\{t\}\times M$ is $t\rho(X_t)(f_t)$, where for any $f\in\mathcal{C}^\infty(M\times[0,\infty))$ we denote by $f_t\in\mathcal{C}^\infty(M)$ the restriction of f to $\{t\}\times M\equiv M$.

Observe that $\mathcal{C}^\infty([0,\infty))\subset\mathcal{C}^\infty(M\times[0,\infty))$ is acted upon trivially by $\Gamma(A_t)$ and hence will define a central subalgebra of the universal enveloping algebra $U(A_t)$ of the adiabatic Lie algebroid A_t. Denote by $t\in\mathcal{C}^\infty([0,\infty))$ the identity function.

Theorem 4. *The inverse limit* $\operatorname{proj\,lim} U(A_t)/t^nU(A_t)$ *is a deformation quantization of* $\Gamma(\mathrm{Symm}(A))$*, the algebra of polynomial functions on* A^**. Therefore, it induces a* $*$*-product on the Lie-Poisson space* A^* *in the sense of* [**3**].

Proof. It follows from the PBW theorem for Lie algebroids (Theorem 3) that the inverse limit $\operatorname{proj\,lim} U(A_t)/t^nU(A_t)$ is isomorphic to $\Gamma(\mathrm{Symm}(A))[[t]]$ as a $\mathbb{C}[[t]]$ module via the complete symbol map $\sigma_\nabla=\sigma_{\nabla,\phi}$ defined in Equation (16). Denote by $\{\ ,\ \}'$ the Poisson bracket on A_t^* and identify $\mathcal{C}^\infty(A^*)$ with the subset of functions on A_t^* that do not depend on t. Then $\{f,g\}'=t\{f,g\}$ if f,g are smooth functions on A_t^* and $\{\ ,\ \}$ is the Poisson bracket on A^*.

For any polynomial function f on A^* denote by $q(f)\in U(A_t)$ the element with complete symbol $\sigma_\nabla(q(f))(\xi,t)=f(\xi)$ obtained, as an application of the isomorphism in the PBW theorem for A_t. (We treat τ as the identity, which justifies replacing $\sigma_\nabla\circ\tau$ with σ_∇.) The proof will be complete if we check the following quantization relation

$$q(f)q(g)-q(g)q(f)=tq(\{f,g\})+t^2h,\quad h\in U(A_t). \tag{25}$$

It is actually enough to do so for f and g among a set of generators of the algebra $\Gamma(\mathrm{Symm}(A))$. Choose the set of generators to be the union of $\mathcal{C}^\infty(M)$ and $\Gamma(A)$. Then Equation (25) will obviously be satisfied for f and g in this generating set (with no t^2–term) in view of the definition of the Lie bracket on A_t. □

Remark. When M is a Lie group G and ∇ is the right invariant trivial connection making all right invariant vector fields parallel, this construction, restricted to right invariant differential operators, reduces to the symmetrization correspondence between $U(\mathfrak{g})$ and $S(\mathfrak{g})$ studied by Berezin [**4**] and Gutt [**12**]. See also Rieffel's paper [**35**]. On the other hand, when the Lie algebroid A is the tangent bundle Lie algebroid TP, this construction gives rise to a quantization for the canonical symplectic structure on cotangent bundle T^*M.

The quantization of the Lie-Poisson structure on A^* was investigated by Landsman in terms of Jordan-Lie algebras [**17**]. His quantization axioms are closer to those of Rieffel's strict deformation quantization. It was conjectured in [**17**] that the quantization of A^* is related to the groupoid C^*-algebra of the corresponding groupoid $\mathcal{G}$, and the transitive case was proved in [**18**].

4. Examples.

As anticipated in the introduction, we recover many previously defined classes of operators as pseudodifferential operators on groupoids. We begin by showing that pseudodifferential operators on a manifold, in the classical sense, are obtained as a particular case of our construction. In this section we will consider only operators with coefficients in the trivial line bundle $E = \mathbb{C}$. We include the description of the Lie algebroids associated to each example.

Denote by $\Psi^m_{\text{prop}}(M)$ the space of *properly supported* pseudodifferential operators on a smooth manifold M, and by $\Psi^m_{\text{comp}}(M)$ the subspace of operators with compactly supported Schwartz kernel, regarded as a distribution on $M \times M$.

Example 1. Let M be a smooth manifold and $\mathcal{G} = M \times M$ be the pair groupoid: $\mathcal{G}^{(1)} = M \times M$, $\mathcal{G}^{(0)} = M$, $d(x,y) = y$, $r(x,y) = x$, $(x,y)(y,z) = (x,z)$. According to the definition, a pseudodifferential operator $P \in \Psi^m(\mathcal{G})$ is a uniformly supported invariant *family* of pseudodifferential operators $P = (P_x, x \in M)$ on $M \times \{x\}$. The action by right translation with $g = (x,y)$ identifies $M \times \{x\}$ with $M \times \{y\}$. After we identify all fibers with M, the invariance condition reads $P_x = P_y$ for all x, y in M. This shows that the family $P = (P_x)_{x \in M}$ is constant, and hence reduces to *one* operator P_0 on M. The family $P = (P_x)_{x \in M}$ is uniformly supported if and only if the distribution kernel of P_0 is compactly supported. The family P is properly supported if and only if P_0 is properly supported. If M is not compact, then $P = (P_x)_{x \in M}$ will not be compactly supported unless it vanishes. We obtain $\Psi^m(\mathcal{G}) = \Psi^m_{\text{comp}}(M)$.

In this case, the Lie algebroid $A(\mathcal{G})$ is the tangent bundle TM.

Example 2. If $\mathcal{G}$ has only one unit, i.e. if $\mathcal{G} = G$, a Lie group, then $\Psi^m(\mathcal{G}) \simeq \Psi^m_{\text{prop}}(G)^G$, the algebra of properly supported pseudodifferential operators on G, invariant with respect to right translations. In this example, every invariant properly supported operator is also uniformly supported. Again, there are no nontrivial compactly supported operators unless G is compact.

In this example $A(\mathcal{G})$ is the Lie algebra of G.

We continue with some more elaborate examples.

Example 3. If $\mathcal{G}$ is the holonomy groupoid[3] of a foliation $\mathcal{F}$ on a smooth manifold M, then $\Psi^\infty(\mathcal{G})$ is the algebra of pseudodifferential operators along the leaves of the foliation [**6**, **8**, **28**, **40**]. Suppose for simplicity that the foliation is given by a (right) locally free action of a Lie group G on a manifold M, and that the isotropy representation of G_x, the stabilizer of x, on N_x, the normal space to the orbit through x, is faithful. This is equivalent to the condition that the holonomy of the leaf passing through x be isomorphic to the discrete group G_x. Then the holonomy groupoid of this foliation is the transformation groupoid $\mathcal{G}^{(1)} = M \times G$, $\mathcal{G}^{(0)} = M$, $d(x,g) = xg$, $r(x,g) = x$ and $(x,g)(xg,g') = (x,gg')$. The algebra $\Psi^\infty(\mathcal{G};E)$ consists of families of pseudodifferential operators on G parametrized by M, invariant with respect to the diagonal action of G and with support contained in a set of the form $\{(x,kg,xk,g)\} \subset (M \times G)^2$, where $g \in G$ is arbitrary but $x \in M$ and $k \in G$ belong to compact sets that depend on the family P.

The Lie algebroid $A = A(\mathcal{G})$ is the integrable subbundle of TM corresponding to the foliation $\mathcal{F}$.

Example 4. Let $\mathcal{G}$ be the fundamental groupoid of a compact smooth manifold M with fundamental group $\pi_1(M) = \Gamma$. Recall that if we denote by $\tilde{M}$ a universal covering of M and let Γ act by covering transformations, then $\mathcal{G}^{(0)} = \tilde{M}/\Gamma = M$, $\mathcal{G}^{(1)} = (\tilde{M} \times \tilde{M})/\Gamma$ and d and r are the two projections. Each fiber $\mathcal{G}_x$ can be identified with $\tilde{M}$, uniquely up to the action of an element in Γ. Let $P = (P_x, x \in M)$ be an invariant, uniformly supported, pseudodifferential operator on $\mathcal{G}$. Then each P_x, $x \in M$ is a pseudodifferential operator on $\tilde{M}$. The invariance condition applied to the elements g such that $x = d(g) = r(g)$ implies that each operator P_x is invariant with respect to the action of Γ. This means that we can identify P_x with an operator on $\tilde{M}$ and that the resulting operator does not depend on the identification of $\mathcal{G}_x$ with $\tilde{M}$. Then the invariance condition applied to an arbitrary arrow $g \in \mathcal{G}^{(1)}$ gives that all operators P_x acting on $\tilde{M}$

[3]The holonomy groupoids of some foliations are non-Hausdorff manifolds. We believe that our constructions will extend to this case with the use of the technique in [**7**] (page 564), where the groupoid algebra is generated by continuous functions supported on Hausdorff open sets.

coincide. We obtain $\Psi^m(\mathcal{G}) \simeq \Psi^m_{\text{prop}}(\tilde{M})^\Gamma$, the algebra of properly supported Γ-invariant pseudodifferential operators on the universal covering $\tilde{M}$ of M. An alternative definition of this algebra using crossed products is given in [**29**]. See also [**6**].

The Lie algebroid is TM, as in the first example.

Example 5. Let Γ be a discrete group acting from the right by diffeomorphisms on a smooth compact manifold M. Define $\mathcal{G}$ as follows, $\mathcal{G}^{(0)} = M$, $\mathcal{G}^{(1)} = M \times M \times \Gamma$ with $d(x,y,\gamma) = y\gamma$, $r(x,y,\gamma) = x$ and $(x,y,\gamma)(y\gamma, y', \gamma') = (x, y'\gamma^{-1}, \gamma\gamma')$. Then $\Psi^\infty(\mathcal{G})$ is the algebra generated by Γ and $\Psi^\infty_{\text{prop}}(M)$ acting on $\mathcal{C}^\infty(M) \otimes \mathbb{C}[\Gamma]$, where Γ acts diagonally, $\Psi^\infty_{\text{prop}}(M)$ acts on the first variable, and $\mathbb{C}[\Gamma]$ denotes the set of finite sums of elements in Γ with complex coefficients. This algebra coincides with the crossed product algebra $\Psi^\infty_{\text{prop}}(M) \rtimes \Gamma = \{\sum_{i=0}^n P_i g_i, P_i \in \Psi^\infty_{\text{prop}}(M), g_i \in \Gamma\}$. The regularizing algebra $\Psi^{-\infty}(\mathcal{G})$ is isomorphic to $\Psi^{-\infty}_{\text{prop}}(M) \rtimes \Gamma \simeq \Psi^{-\infty}_{\text{prop}}(M) \otimes \mathbb{C}[\Gamma]$. If we drop the condition that M be compact we obtain $\Psi^\infty(\mathcal{G}) \simeq \Psi^\infty_{\text{comp}}(M) \rtimes \Gamma$.

In general, if a discrete group Γ acts on a groupoid $\mathcal{G}_0$, then

$$\Psi^\infty_{\text{prop}}(\mathcal{G}_0 \rtimes \Gamma) \simeq \Psi^\infty_{\text{prop}}(\mathcal{G}_0) \rtimes \Gamma.$$

This construction does not change the Lie algebroid.

In the following example we realize the algebra of families of operators in $\Psi^m(\mathcal{G})$ parametrized by a compact space B as the algebra of pseudodifferential operators on the product groupoid $\mathcal{G} \times B$. This example shows that our class of operators on groupoids is closed under formation of families of operators.

Example 6. If B is a compact manifold with corners, define $\mathcal{G} \times B$ by $(\mathcal{G} \times B)^{(0)} = \mathcal{G}^{(0)} \times B$, $(\mathcal{G} \times B)^{(1)} = \mathcal{G}^{(1)} \times B$ with the structural maps preserving the B-component. Then $\Psi^m(\mathcal{G} \times B)$ contains $\Psi^m(\mathcal{G}) \otimes \mathcal{C}^\infty(B)$ as a dense subset in the sense that $\Psi^{m-1}(\mathcal{G}) \otimes \mathcal{C}^\infty(B) = \Psi^m(\mathcal{G}) \otimes \mathcal{C}^\infty(B) \cap \Psi^{m-1}(\mathcal{G} \times B)$ and $\Psi^m(\mathcal{G}) \otimes \mathcal{C}^\infty(B)/\Psi^{m-1}(\mathcal{G}) \otimes \mathcal{C}^\infty(B)$ is dense in $\Psi^m(\mathcal{G} \times B)/\Psi^{m-1}(\mathcal{G} \times B)$ in the corresponding Frechet topology (defined by the isomorphism of Theorem 8). It follows that $\Psi^m(\mathcal{G} \times B)$ consists of smooth families of operators in $\Psi^m(\mathcal{G})$ parametrized by B, see [**2**], page 122 and after, where families of pseudodifferential operators are discussed.

We obtain that $A(\mathcal{G} \times B)$ is the pull back of A to $\mathcal{G}^{(0)} \times B$.

The following example generalizes the tangent groupoid of Connes; here we closely follow [**6**, II,5]. The groupoid defined below also appears in [**37**] and is related to the notion of explosion of manifolds.

Example 7. The adiabatic groupoid $\mathcal{G}_{\text{adb}}$ associated to $\mathcal{G}$ is defined as follows. The space of units is $\mathcal{G}^{(0)}_{\text{adb}} = [0,\infty) \times \mathcal{G}^{(0)}$ with the product manifold structure. The set of arrows $\mathcal{G}^{(1)}_{\text{adb}}$ is defined to be the disjoint union $A(\mathcal{G}) \cup$

$(0, \infty) \times \mathcal{G}^{(1)}$, and $d(t, g) = (t, d(g))$, $r(t, g) = (t, r(g))$ if $t > 0$, $d(v) = r(v) = (0, x)$ if $v \in T_x\mathcal{G}_x$. The composition is $\mu(\gamma, \gamma') = (t, gg')$ if $\gamma = (t, g)$ and $\gamma' = (t, g')$ for $t > 0$ (necessarily the same t!) and $\mu(v, v') = v + v'$ if $v, v' \in T_x\mathcal{G}_x$.

The smooth structure on the set of arrows is the product structure for $t > 0$. In order to define a coordinate chart at a point $v \in T_x\mathcal{G}_x$ choose first a coordinate system $\psi : U = U_1 \times U_2 \to \mathcal{G}^{(1)}$, $U_1 \subset \mathbb{R}^p$ and $U_2 \subset \mathbb{R}^n$ being open sets containing the origin, U_2 convex, with the following properties: $\psi(0, 0) = x \in \mathcal{G}^{(0)} \subset \mathcal{G}^{(1)}$, $d(\psi(s, y_1)) = d(\psi(s, y_2)) = \phi(s)$ and $\psi(U) \cap \mathcal{G}^{(0)} = \psi(U_1 \times \{0\})$. Here $\phi : U_1 \to \mathcal{G}^{(0)}$ is a coordinate chart of x in $\mathcal{G}^{(0)}$. We identify, using the differential $\mathrm{D}_2\psi$ of the map ψ, the vector space $\{s\} \times \mathbb{R}^n$ and the tangent space $T_{\phi(s)}\mathcal{G}_{\phi(s)} = A_{\phi(s)}(\mathcal{G})$. We obtain then coordinate charts $\psi_\epsilon : [0, \epsilon) \times U_1 \times \epsilon^{-1}U_2 \to \mathcal{G}^{(1)}$, $\psi_\epsilon(0, s, y) = (0, (\mathrm{D}_2\psi)(s, y)) \in T_{\phi(s)}\mathcal{G}_{\phi(s)} = A_{\phi(s)}(\mathcal{G})$ and $\psi_\epsilon(t, s, y) = (t, \psi(s, ty)) \in (0, 1) \times \mathcal{G}^{(1)}$. For ϵ very small the range of ψ_ϵ will contain v.

For $\mathcal{G} = M \times M$ as in the first example the groupoid $\mathcal{G}_{\mathrm{adb}}$ is the tangent groupoid defined by Connes, and the algebra of pseudodifferential operators is the algebra of asymptotic pseudodifferential operators [**39**]. In general an operator P in $\Psi^m(\mathcal{G}_{\mathrm{adb}})$ will restrict to an adiabatic family $P = (P_{t,x}, t > 0, x \in \mathcal{G}^{(0)})$, which will have an "adiabatic limit" at $t = 0$ given by the operator P at $t = 0$.

The Lie algebroid of $\mathcal{G}_{\mathrm{adb}}$ is the adiabatic Lie algebroid associated to $A(\mathcal{G})$, $A(\mathcal{G}_{\mathrm{adb}}) = A(\mathcal{G})_t$ (using the notation of Theorem 4). This gives a procedure for integrating adiabatic Lie algebroids. Using pseudodifferential operators on the adiabatic groupoid we obtain an explicit quantization of symbols on A^* generalizing Theorem 4. The proof proceeds exactly in the same way.

Theorem 5. *The inverse limit* $\operatorname{proj\,lim} \Psi^\infty(\mathcal{G}_{\mathrm{adb}})/t^n\Psi^\infty(\mathcal{G}_{\mathrm{adb}})$ *is a deformation quantization of the commutative algebra* $\mathcal{S}^\infty_{\mathrm{cl}}(A^*(\mathcal{G}))$ *of classical symbols.*

The space $\mathcal{S}^\infty_{\mathrm{cl}}(A^*(\mathcal{G}))$ appearing in the statement of the theorem above is the union of all symbol spaces $\mathcal{S}^m_{\mathrm{cl}}(A^*(\mathcal{G}))$ and is a commutative algebra under pointwise multiplication.

Of course Theorem 4 provides us with a $*$-product whose multiplication is given by differential operators, and hence this $*$-product extends to all smooth functions (even to functions defined on open subsets). The usefulness of the theorem above is that it gives in principle a *nonperturbative* (i.e. not just formal) deformation quantization, close in spirit to that of *strict* deformation quantization introduced by Rieffel [**34**].

Example 8. This example provides a treatment in our settings of the b- and c-calculi defined by Melrose [**21**, **22**, **25**, **26**] on a manifold with boundary M.

Define first a groupoid $\mathcal{G}_\phi(M)$ associated to M and an increasing diffeomorphism $\phi : \mathbb{R} \to (0,\infty)$ as follows. If $M = [0,\infty)$ the action by translation of $\mathbb{R}$ on itself extends to an action on M fixing 0, not smooth in general, defined using the isomorphism ϕ. Define $\mathcal{G}_\phi(M)$ to be the transformation groupoid associated this action of $\mathbb{R}$ on M. If $M = [0,1)$, then $\mathcal{G}_\phi(M)$ is defined to be the reduction $\mathcal{G}_\phi(M) = \mathcal{G}_\phi([0,\infty)) \cap d^{-1}(M) \cap r^{-1}(M)$ of $\mathcal{G}_\phi([0,\infty))$ to $[0,1)$.

Suppose next that $M = \partial M \times [0,1)$. We then define $\mathcal{G}_\phi(M) = \mathcal{G}_\phi([0,1)) \times (\partial M \times \partial M)$, where $\partial M \times \partial M$ is the pair groupoid of ∂M considered in Example 1. For an arbitrary manifold with boundary M write $M = M_0 \cup U$, where $U = M \setminus \partial M$ and M_0 is diffeomorphic to $\partial M \times [0,1)$. (Our construction will depend on this diffeomorphism.) Then we define $\mathcal{G}_\phi^{(0)}(M) = M$ and $\mathcal{G}_\phi^{(1)}(M) = \mathcal{G}_\phi^{(1)}(M_0) \cup (U \times U)$ with the induced operations.

If $\phi(t) = e^t$ or $\phi(t) = -t^{-1}$ (for $t << 0$), then $\mathcal{G} = \mathcal{G}_\phi$ will be an almost differentiable groupoid and we obtain $\Psi^m(\mathcal{G}) \subset \Psi_b(M)$ in the first case and $\Psi^m(\mathcal{G}) \subset \Psi_c(M)$ in the second case. The first groupoid does not depend on any choices.

5. Distribution kernels.

In this section we characterize the reduced (or convolution) distribution kernels of operators in $\Psi^m(\mathcal{G}; E)$ following [21] (see also [14]) as compactly supported distributions on $\mathcal{G}$, conormal to the set of units $\mathcal{G}^{(0)}$.

Denote by $\mathrm{END}_{\mathcal{G}}(E)$ the bundle $\mathrm{Hom}(d^*(E), r^*(E)) = r^*(E) \otimes d^*(E)'$ on $\mathcal{G}^{(1)}$, where V' denotes as usual the dual of the vector bundle V. Using the relations $d \circ \iota = r$ and $r \circ \iota = d$ we see that $\mathrm{END}_{\mathcal{G}}(E)$ satisfies

$$\iota^*(\mathrm{END}_{\mathcal{G}}(E)) \simeq d^*(E) \otimes r^*(E)' \simeq \mathrm{END}_{\mathcal{G}}(E)'. \tag{26}$$

We define a convolution product on the space $\mathcal{C}_c^\infty(\mathcal{G}^{(1)}, \mathrm{END}_{\mathcal{G}}(E) \otimes d^*(\mathcal{D}))$ of compactly supported smooth sections of the bundle $\mathrm{END}_{\mathcal{G}}(E) \otimes d^*(\mathcal{D})$ by the formula

$$f_1 * f_2(g) = \int_{\{(h_1,h_2),h_1h_2=g\}} f_1(h_1)f_2(h_2)\ . \tag{27}$$

The multiplication on the right hand side is the composition of homomorphisms giving a linear map

$$\begin{gathered} \mathrm{Hom}(E_{d(h_1)}, E_{r(h_1)}) \otimes \mathrm{Hom}(E_{d(h_2)}, E_{r(h_2)}) \otimes \mathcal{D}_{d(h_1)} \otimes \mathcal{D}_{d(h_2)} \longrightarrow \\ \mathrm{Hom}(E_{d(g)}, E_{r(g)}) \otimes \mathcal{D}_{d(h_1)} \otimes \mathcal{D}_{d(h_2)} \\ f_1(h_1) \otimes f_2(h_2) \longrightarrow f_1(h_1)f_2(h_2), \end{gathered}$$

defined since $d(h_1) = r(h_2)$. To see that the integration is defined we parametrize the set $\{(h_1,h_2), h_1h_2 = g\}$ as $\{(gh^{-1}, h), h \in \mathcal{G}_{d(g)}\}$, which shows that this set is a smooth manifold, and notice that we can invariantly define the integration with respect to h taking advantage of the 1-density factor $\mathcal{D}_{d(h_1)} = \mathcal{D}_{r(h)} = (\Omega_d)_h$. If we choose a hermitian metric on $\mathcal{D}^{-1/2} \otimes E$, we obtain a conjugate–linear involution (making $\mathcal{C}_c^\infty(\mathcal{G}^{(1)}, \mathrm{END}_{\mathcal{G}}(E) \otimes d^*(\mathcal{D}))$ into a $*$-algebra).

Consider an operator $P = (P_x, x \in \mathcal{G}^{(0)}) \in \Psi^{-\infty}(\mathcal{G}; E)$ and let k_x be the distribution kernel of P_x, a smooth section $k_x \in \mathcal{C}^\infty(\mathcal{G}_x \times \mathcal{G}_x; r_1^*(E) \otimes r_2^*(E)' \otimes \Omega_2)$, using the notation $\Omega_2 = p_2^*(\Omega_d) = r_2^*(\mathcal{D})$ of (11). We define the reduced distribution kernel k_P of the smoothing operator P by

$$k_P(g) = k_{d(g)}(g, d(g)) \in E_{r(g)} \otimes E'_{d(g)} \otimes \mathcal{D}_{d(g)}\,. \tag{28}$$

This definition will be later extended to all of $\Psi^\infty(\mathcal{G}; E)$.

The following theorem is one of the main reasons we consider *uniformly* supported operators.

Theorem 6. *The reduced kernel map $P \to k_P$ (28) defines an isomorphism of the residual ideal $\Psi^{-\infty}(\mathcal{G}; E)$ with the convolution algebra $\mathcal{C}_c^\infty(\mathcal{G}^{(1)}, \mathrm{END}_{\mathcal{G}}(E) \otimes d^*(\mathcal{D}))$.*

Proof. Let P and k_x be as above. We know from Lemma 1 that the collection of all sections k_x defines a *smooth* section of $r_1^*(E) \times r_2^*(E)' \otimes \Omega_2$ over the manifold $\{(g_1, g_2), d(g_1) = d(g_2)\}$. The relation $P_{r(g)} U_g = U_g P_{d(g)}$ gives the invariance relation $k_{r(g)}(h', h) = k_{d(g)}(h'g, hg) \in E_{r(h')} \otimes E'_{r(h)} \otimes \mathcal{D}_{r(h)}$ for all arrows $g \in \mathcal{G}^{(1)}$, and $h, h' \in \mathcal{G}_{r(g)}$. It follows that $k_{d(h)}(h', h) = k_{r(h)}(h'h^{-1}, r(h)) = k_P(h'h^{-1})$. The section k_P is well defined, smooth and completely determines all kernels k_x and hence also the operator P. Moreover the section k_P has compact support because $\mathrm{supp}(k_P) = \mathrm{supp}_\mu(P) = \mu \circ (id \times \iota)\big(\overline{\cup_x \mathrm{supp}(k_x)}\big)$ and the reduced support $\mathrm{supp}_\mu(P)$ of P is compact since P is uniformly supported. The distribution kernel k_x^{PQ} of the product $P_x Q_x$ of two operators $P_x, Q_x \in \Psi^{-\infty}(\mathcal{G}_x)$ is

$$k_x^{PQ}(g, g'') = \int_{\mathcal{G}_x} k_x^P(g, g') k_x^Q(g', g'') dg',$$

where k_x^P and k_x^Q are the distribution kernels of P_x and, respectively, Q_x. From this, taking into account the definitions of k_{PQ}, k_P and k_Q, we obtain

$$k_{PQ}(g) = \int_{\mathcal{G}_x} k_P(g{g'}^{-1}) k_Q(g') dg' \ .$$

This means that $k_{PQ} = k_P * k_Q$ and hence the map $P \to k_P$ establishes the desired isomorphism. □

We will now use duality to extend the definition of the reduced distribution kernel to any operator $P \in \Psi^\infty(\mathcal{G}; E)$. Let $\mathcal{L} = \Omega_{\mathcal{G}^{(0)}}$ be the line bundle of 1-densities on $\mathcal{G}^{(0)}$ and $\mathcal{D} = \Omega_d|_{\mathcal{G}^{(0)}}$ be the bundle of vertical 1-densities as above. Define

$$\Psi^{-\infty}(\mathcal{G}; E)_{\mathcal{L}} = \Psi^{-\infty}(\mathcal{G}; E) \otimes_{\mathcal{C}^\infty(\mathcal{G}^{(0)})} \mathcal{C}^\infty(\mathcal{G}^{(0)}, \mathcal{D}^{-1} \otimes \mathcal{L}) \simeq$$
$$\Psi^{-\infty}(\mathcal{G}; E) \otimes_{\mathcal{C}^\infty(\mathcal{G}^{(0)})} \mathcal{C}^\infty(\mathcal{G}^{(0)}, \mathcal{D}^{-1}) \otimes_{\mathcal{C}^\infty(\mathcal{G}^{(0)})} \mathcal{C}^\infty(\mathcal{G}^{(0)}, \mathcal{L}),$$

where the tensor products are defined using the inclusion $\mathcal{C}^\infty(\mathcal{G}^{(0)}) \subset \Psi^\infty(\mathcal{G}; E)$. We note that the bundle $\mathcal{L}$ plays an important role in connection with the modular class of a groupoid [**11**], since it carries a natural representation of the groupoid.

The relation $k_{Pf}(g) = k_P(g)f(d(g))$ for $f \in \mathcal{C}^\infty(\mathcal{G}^{(0)})$ and $P \in \Psi^{-\infty}(\mathcal{G}; E)$ give using Theorem 6 the isomorphism

$$\Psi^{-\infty}(\mathcal{G}; E)_{\mathcal{L}} \simeq \mathcal{C}_c^\infty(\mathcal{G}^{(1)}, \mathrm{END}_{\mathcal{G}}(E) \otimes d^*(\mathcal{L})). \tag{29}$$

The space $\Psi^{-\infty}(\mathcal{G}; E)_{\mathcal{L}}$ comes equipped with a natural linear functional T such that, if $P_0 \in \Psi^{-\infty}(\mathcal{G}; E)$, $\xi \in \mathcal{C}^\infty(\mathcal{G}^{(0)}, \mathcal{D}^{-1})$ and $\nu \in \mathcal{C}^\infty(\mathcal{G}^{(0)}, \mathcal{L})$, then

$$\mathrm{T}(P_0 \otimes \xi \otimes \nu) = \int_{\mathcal{G}^{(0)}} tr(k_{P_0}(x)\xi(x))d\nu(x)$$

defined by integrating the function $tr(k_{P_0}(x)\xi(x))$ with respect to the 1-density (i.e. measure) ν. An operator $P \in \Psi^m(\mathcal{G}; E)$ defines a continuous linear functional (i.e. distribution) $k_P^\iota : \Psi^{-\infty}(\mathcal{G}; E)_{\mathcal{L}} \to \mathbb{C}$ by the formula $k_P^\iota(P_0 \otimes \xi \otimes \nu) = \mathrm{T}(PP_0 \otimes \xi \otimes \nu)$. It is easy to see using Equation (26) that the map $f \to \tilde{f} = f \circ \iota$, $\iota(g) = g^{-1}$, establishes isomorphisms

$$\begin{aligned} \Phi : \mathcal{C}_c^\infty(\mathcal{G}^{(1)}, \mathrm{END}_{\mathcal{G}}(E) \otimes d^*(\mathcal{L})) &\xrightarrow{\iota^*} \mathcal{C}_c^\infty(\mathcal{G}^{(1)}, \mathrm{END}_{\mathcal{G}}(E)' \otimes r^*(\mathcal{L})) \\ &\simeq \mathcal{C}_c^\infty(\mathcal{G}^{(1)}, (\mathrm{END}_{\mathcal{G}}(E) \otimes d^*(\mathcal{L}))' \otimes d^*(\mathcal{L}) \otimes r^*(\mathcal{L})) \\ &\simeq \mathcal{C}_c^\infty(\mathcal{G}^{(1)}, (\mathrm{END}_{\mathcal{G}}(E) \otimes d^*(\mathcal{L}))' \otimes \Omega_{\mathcal{G}}) \end{aligned} \tag{30}$$

whose composition we denote by Φ, so that $\Phi(P_0 \otimes \xi \otimes \nu) = (k_{P_0}\xi\nu) \circ \iota = \iota^*(k_{P_0}\xi\nu)$. We obtain in this way from k_P^ι a distribution $k_P \in \mathcal{C}^{-\infty}(\mathcal{G}^{(1)};$ $\mathrm{END}_{\mathcal{G}}(E) \otimes d^*(\mathcal{D}))$ defined by the formula

$$\langle k_P, f\rangle = k_P^\iota(\Phi^{-1}(f)). \tag{31}$$

An other way of writing the formula above is

$$\langle k_P, \iota^*(k_{P_0}\xi\nu)\rangle = \mathrm{T}(PP_0 \otimes \xi \otimes \nu) = \int_{\mathcal{G}^{(0)}} tr(k_{PP_0}(x)\xi(x))d\nu(x). \tag{32}$$

Proposition 6. *If $P \in \Psi^{-\infty}(\mathcal{G}; E)$ is a regularizing operator, then the kernels k_P defined in Equations (28) and (31) coincide.*

Proof. To make a distinction for the purpose of this proof, denote by k_P^{dist} the distribution defined by (31). Let ν be a smooth section of $\mathcal{L}$, ξ a smooth section of $\mathcal{D}^{-1}$ and $P, P_0 \in \Psi^{-\infty}(\mathcal{G}; E)$. Using Equation (32) we obtain

$$\begin{aligned}
\left\langle k_P^{\mathrm{dist}}, \iota^*(k_{P_0}\xi\nu)\right\rangle &= \mathrm{T}(PP_0 \otimes \xi \otimes \nu) \\
&= \int_{\mathcal{G}^{(0)}} \mathrm{tr}(k_{PP_0}(x)\xi(x))d\nu(x) \\
&= \int_{\mathcal{G}^{(0)}} \left(\int_{\mathcal{G}_x} \mathrm{tr}\big(k_P(h^{-1})k_{P_0}(h)\xi(x)\big)\right) d\nu(x) \\
&= \langle k_P \circ \iota, k_{P_0}\xi\nu\rangle = \langle k_P, \iota^*(k_{P_0}\xi\nu)\rangle.
\end{aligned}$$

□

Definition 9. The distribution $k_P \in \mathcal{C}^{-\infty}(\mathcal{G}^{(1)}; \mathrm{END}_{\mathcal{G}}(E) \otimes d^*(\mathcal{D}))$, defined for any operator $P \in \Psi^m(\mathcal{G}; E)$ by Equation (31) will be called the reduced (or convolution) distribution kernel of P, or simply the reduced kernel of P, and will be denoted k_P.

We now relate the action of $\Psi^\infty(\mathcal{G}; E)$ by multiplication on $\Psi^{-\infty}(\mathcal{G}; E)$, respectively on $\Psi^{-\infty}(\mathcal{G}; E)_{\mathcal{L}}$, to that on $\mathcal{C}_c^\infty(\mathcal{G}, r(E))$.

$$\Psi^{-\infty}(\mathcal{G}; E) \simeq \mathcal{C}_c^\infty(\mathcal{G}^{(1)}; r^*(E)) \otimes_{\mathcal{C}^\infty(\mathcal{G}^{(0)})} \Gamma(E') \otimes_{\mathcal{C}^\infty(\mathcal{G}^{(0)})} \Gamma(\mathcal{D}) \tag{33}$$

$$\Psi^{-\infty}(\mathcal{G}; E)_{\mathcal{L}} \simeq \mathcal{C}_c^\infty(\mathcal{G}^{(1)}; r^*(E)) \otimes_{\mathcal{C}^\infty(\mathcal{G}^{(0)})} \Gamma(E') \otimes_{\mathcal{C}^\infty(\mathcal{G}^{(0)})} \Gamma(\mathcal{L}) \tag{34}$$

such that the left action by multiplication of $\Psi^\infty(\mathcal{G}; E)$ on $\Psi^{-\infty}(\mathcal{G}; E)$ becomes $P(f \otimes \eta \otimes \xi) = Pf \otimes \eta \otimes \xi$, where η is a smooth section of E' and ξ is a smooth section of $\mathcal{D}$ or $\mathcal{L}$. Moreover the kernel of $P_0 = f \otimes \eta \otimes \xi$ is $k_{P_0}(g) = f(g) \otimes \eta(d(g))\xi(d(g))$. Thus in order to define the distribution k_P, for arbitrary P, it is enough to compute $\mathrm{T}(Pf_0 \otimes \eta \otimes \nu)$, where ν is a density.

Fix a unit x and choose a coordinate chart $\phi : U_0 \to U \subset \mathcal{G}^{(0)}$, where U_0 is an open subset of $\mathbb{R}^k$ containing 0, $k = \dim \mathcal{G}^{(0)}$ and $\phi(0) = x$. By decreasing U_0 if necessary we can assume that the tangent space $T\mathcal{G}^{(0)}$ is trivialized over U. Consider the diffeomorphism $\exp_\nabla : V_0 \to V \subset \mathcal{G}^{(1)}$ associated to a right invariant connection ∇ as in (14) and (15), where $V_0 \subset A(\mathcal{G})$ is an open neighborhood of the zero section. It maps the zero section of $A(\mathcal{G})$ to $\mathcal{G}^{(0)}$. Choose a connection on E, which lifts to an invariant connection on $r^*(E)$. By decreasing V if necessary and using the invariant connection on $r^*(E)$ we obtain canonical trivializations of $r^*(E)$ on each fiber $V \cap \mathcal{G}_x$. Denote by $\theta_h : E_{r(h)} \otimes E'_{d(h)} \to \mathrm{End}(E_x)$ the isomorphism induced by the connection ∇' (defined using parallel transport along the geodesics of ∇), where $x = d(h)$ and h is in V. Decreasing further V and U we can assume that E is trivialized over $d^{-1}(U) \cap V$ and that ϕ and $\exp_\nabla$ give a fiber preserving diffeomorphism $\psi : U_0 \times W \to d^{-1}(U) \cap V$, where $W \subset \mathbb{R}^n$ is an open set, identified with an open neighborhood of the zero section in $T_x\mathcal{G}^{(0)}$.

The diffeomorphism ψ we have just constructed satisfies $d(\psi(s,y)) = \phi(s)$. The maps ψ and θ_h yield isomorphisms

$$\begin{aligned}(35)\qquad &\mathcal{C}^{-\infty}(d^{-1}(U)\cap V, \mathrm{END}_{\mathcal{G}}(E)\otimes d^*(\mathcal{D}))\\ &\simeq \mathcal{C}^{-\infty}(U_0\times W, \psi^*(\mathrm{END}_{\mathcal{G}}(E)\otimes d^*(\mathcal{D})))\\ &\simeq \mathcal{C}^{-\infty}(U_0\times W, E_x\otimes E_x')\end{aligned}$$

whose composition is denoted Θ_ψ.

Next theorem describes the reduced distribution kernels k_P of operators P in $\Psi^m(\mathcal{G};E)$. We use the notation introduced above.

Theorem 7. *For any operator $P = (P_x, x\in\mathcal{G}^{(0)}) \in \Psi^m(\mathcal{G};E)$ the reduced distribution kernel k_P satisfies:*

(i) *If $\psi : U_0\times W \to V_1 \subset V$, $W\subset\mathbb{R}^n$ open, is a diffeomorphism satisfying $\psi(s,0) = d(\psi(s,y))$, then there exists a symbol $a_P \in \mathcal{S}^m_{\mathrm{cl}}(U_0\times\mathbb{R}^n;\mathrm{End}(E_x))$, such that $k_P\circ\iota = \Theta_\psi^{-1}(k)$ on V_1, where k is the distribution*

$$k(s,y) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{-iy\cdot\zeta}a_P(s,\zeta)d\zeta \in \mathrm{End}(E_x),$$

the integral being an oscillatory integral. Moreover, after suitable identifications, a_P is a representative of the principal symbol of P.

(ii) *The singular support of k_P is contained in $\mathcal{G}^{(0)}$.*

(iii) *The support of k_P is compact, more precisely* $\mathrm{supp}(k_P) = \mathrm{supp}_\mu(P)$.

(iv) *For every distribution $k\in\mathcal{C}^{-\infty}(\mathcal{G};E_0)$, satisfying the three conditions above, there exists $P\in\Psi^m(\mathcal{G};E)$ such that $k = k_P$.*

Note that $k_P\circ\iota$, $\iota^*(k_P)$ and k_P^ι all denote the same distribution.

Proof. Write $\phi(s)$ for $\psi(s,0) = d(\psi(s,y))$. According to Definitions 6 and 7, there exists a classical symbol $a\in\mathcal{S}^m_{\mathrm{cl}}(U_0\times T^*W;\mathrm{End}(E_x))$ such that $P_{\phi(s)} = a(s,y,D_y)$ (modulo regularizing operators) on $\mathcal{G}_{\phi(s)}\cap V_1 \simeq W$.

Let $P_0\in\Psi^{-\infty}(\mathcal{G};E)$, $\xi\in\mathcal{C}^\infty(\mathcal{G}^{(0)},\mathcal{D}^{-1})$ and $\nu\in\mathcal{C}^\infty(\mathcal{G}^{(0)},\mathcal{L})$. Assume, using the isomorphisms (33) and (34), that $P_0 = f_0\otimes\eta$, where $\eta\in\Gamma(E') = \mathcal{C}^\infty(\mathcal{G}^{(0)},E')$ and $f_0\in\mathcal{C}^\infty(\mathcal{G},r^*(E)\otimes\Omega_d)$, so that $f_0\xi$ is a section of $\mathcal{C}^\infty_{\mathrm{c}}(\mathcal{G},r^*(E))$. Then we have

$$tr(k_{PP_0}(x)\xi(x)) = \eta(P_x(f_0\xi|_{\mathcal{G}_x})(x)).$$

Suppose f_0 is supported in V_1, and denote by f_s the section of E_x that corresponds to $f_0\xi|_{\mathcal{G}_{\phi(s)}}$ under the diffeomorphism $\mathcal{G}_{\phi(s)}\cap V_1\simeq\{s\}\times W = W$

induced by ψ. We then have

$$\begin{aligned}
\langle \iota^*(k_P), k_{P_0}\xi\nu\rangle &= \langle k_P, \iota^*(k_{P_0}\xi\nu)\rangle = \mathrm{T}(PP_0 \otimes \xi \otimes \nu)\\
&= \int_{\mathcal{G}^{(0)}} \mathrm{tr}(k_{PP_0}(x)\xi(x))d\nu(x)\\
&= \int_{\mathcal{G}^{(0)}\cap U} \eta\big(P_x(f_0\xi|_{\mathcal{G}_x})(x)\big)d\nu(x)\\
&= \int_{U_0} \eta\left(\int_{\mathbb{R}^n}\int_W e^{-iy\cdot\zeta}a(s,0,\zeta)f_s(y)dyd\zeta\right)d\nu(s)\\
&= \int_{U_0}\int_{\mathbb{R}^n}\int_W e^{-iy\cdot\zeta}\mathrm{tr}\big(a(s,0,\zeta)f_s(y)\otimes\eta\big)dyd\zeta d\nu(s)\\
&= \int_{U_0\times W} \mathrm{tr}\left(f_s(y)\otimes\eta\int_{\mathbb{R}^n} e^{-iy\cdot\zeta}a(s,0,\zeta)d\zeta\right)dyd\nu(s),
\end{aligned}$$

where the first integral is really a pairing between the distribution k obtained from (i) for $a_P(s,\zeta) = a(s,0,\zeta)$, and the smooth section $f_s \otimes \eta \otimes \nu$. Since $\mathrm{End}(E_x)$ is canonically its own dual this shows that the distribution k_P is the conormal distribution to $\mathcal{G}^{(0)}$ given by (i).

To prove (iii) and (iv) observe that k_x is the restriction to $\mathcal{G}_x \times \mathcal{G}_x$ of the distribution $\mu_1^*(k_P)$, where $\mu_1(h', h) = h'h^{-1}$ and the distribution $\mu_1^*(k_P)$ is defined by $\langle \mu_1^*(k_P), f\rangle = \langle k_P(g), \int_{h_1h_2=g} f(h_1,h_2)\rangle$. Then we can define P_x by its distribution kernel k_x. From (i) it follows that k_x is conormal to the diagonal and hence P_x is a pseudodifferential operator.

In order to check (ii) fix $g \notin \mathcal{G}^{(0)}$ and let φ be a smooth cut-off function, $\varphi = 1$ in a neighborhood of $\mathcal{G}^{(0)}$, $\varphi = 0$ in a neighborhood of g. Consider again the distribution $\mu_1^*((1-\varphi)k_P) = (1-\varphi\circ\mu_1)\mu_1^*(k_P)$. Its restriction to $\mathcal{G}_x$ is $(1-\varphi\circ\mu_1)k_x$, which is smooth since the singular support of k_x (= the distribution kernel of P_x) is contained in the diagonal of $\mathcal{G}_x \times \mathcal{G}_x$, and $1-\varphi\circ\mu_1$ vanishes there. It follows that $\mu_1^*((1-\varphi)k_P)$ is smooth and hence $(1-\varphi)k_P$ is also smooth. □

Corollary 1. *The distribution k_P is conormal at $\mathcal{G}^{(0)}$ and smooth everywhere else. In particular, the wave-front set of k_P is a subset of the annihilator of $T\mathcal{G}^{(0)}$: $WF(k_P) \subset (T\mathcal{G}/T\mathcal{G}^{(0)})^* \subset T^*\mathcal{G}|_{\mathcal{G}^{(0)}}$.*

Proof. This is a standard consequence of (i) and (ii) above, see [**14**], Section 12.2. □

We remark that $(T\mathcal{G}/T\mathcal{G}^{(0)})^*$ is naturally identified with $A^*(\mathcal{G})$. Denote by $\mathcal{S}^m_c(A^*(\mathcal{G}); \mathrm{End}(E)) \subset \mathcal{S}^m_{\mathrm{cl}}(A^*(\mathcal{G}); \mathrm{End}(E))$ the space of classical symbols with support in a set of the form $\pi^{-1}(K)$, where $\pi : A^*(\mathcal{G}) \to \mathcal{G}^{(0)}$ is the projection and $K \subset \mathcal{G}^{(0)}$ is a compact subset.

Corollary 2. *Let V be a neighborhood of $\mathcal{G}^{(0)}$ in $\mathcal{G}$. Then any $P \in \Psi^m(\mathcal{G}; E)$ can be written as $P = P_1 + P_2$, where P_1 has reduced support* $\mathrm{supp}_\mu(P_1)$ *contained in V and $P_2 \in \Psi^{-\infty}(\mathcal{G}; E)$.*

Proof. Let ϕ be a smooth cut-off function, equal to 1 in a neighborhood of $\mathcal{G}^{(0)}$ and with support in V. Define $P_2 \in \Psi^{-\infty}(\mathcal{G}; E)$ by $k_{P_2} = k_P(1-\phi)$. This is possible using Theorem 6 since by (ii) of the theorem above $k_P(1-\phi)$ is a smooth compactly supported section of an appropriate bundle. Then $P_1 = P - P_2$ and P_2 satisfy the requirements of the statement. □

Theorem 8. *The principal symbol map σ_m in Equation (17) is onto; hence it establishes an isomorphism*

$$\Psi^m(\mathcal{G}; E)/\Psi^{m-1}(\mathcal{G}; E) \simeq \mathcal{S}_c^m(A^*(\mathcal{G}); \mathrm{End}(E))/\mathcal{S}_c^{m-1}(A^*(\mathcal{G}); \mathrm{End}(E))$$

for any m.

Proof. We only need to prove that σ_m is onto. If follows from the proof of Theorem 7 that $\sigma_m(P)$ is the class of the symbol a_P appearing in the equation in (i). Given a symbol $a \in \mathcal{S}_c^m(A^*(\mathcal{G}); \mathrm{End}(E))$ the equation in (i) defines a distribution k_0 in a small neighborhood of $\mathcal{G}^{(0)}$ in $\mathcal{G}$. Using a smooth cut-off function we obtain a distribution k on $\mathcal{G}$ that coincides with k in a neighborhood of $\mathcal{G}^{(0)}$ and is smooth outside $\mathcal{G}^{(0)}$. From (iv) we conclude that there exists an operator P with $k_P = k$, which will then necessarily satisfy $\sigma_m(P) = a + \mathcal{S}_c^{m-1}(A^*(\mathcal{G}); \mathrm{End}(E))$. □

6. The action on sections of E.

In this section we define a natural action of $\Psi^m(\mathcal{G}; E)$ on sections of E over $\mathcal{G}^{(0)}$, thus generalizing the action of classical pseudodifferential operators on functions.

Let ϕ be a smooth section of E over $\mathcal{G}^{(0)}$. Define

$$\tilde{\phi} \in \mathcal{C}^\infty(\mathcal{G}^{(1)}, r^*(E)),\ \tilde{\phi}(g) = \phi(r(g)). \tag{36}$$

Lemma 8. *If $P = (P_x, x \in \mathcal{G}^{(0)})$ belongs to $\Psi^\infty(\mathcal{G}; E)$, then for any section ϕ in $\mathcal{C}^\infty(\mathcal{G}^{(0)}, E)$ there exists a unique section $\psi \in \mathcal{C}^\infty(\mathcal{G}^{(0)}, E)$ such that $P\tilde{\phi} = \tilde{\psi}$.*

Proof. Observe first that given a section γ of $r^*(E)$ over $\mathcal{G}^{(1)}$ we can find a section ϕ of E over $\mathcal{G}^{(0)}$ such that $f = \tilde{\phi}$ if and only if $f(g'g) = f(g')$ for all g and g', i.e. if and only if

$$U_g f_x = f_y, \text{ for all } g,\, x,\, y \text{ such that } x = d(g) \text{ and } y = r(g). \tag{37}$$

We then have

$$U_g\tilde{\phi}_x = \tilde{\phi}_y \Rightarrow P_y U_g \tilde{\phi}_x = P_y\tilde{\phi}_y \Rightarrow U_g P_x \tilde{\phi}_x = P_y \tilde{\phi}_y \Rightarrow U_g(P\tilde{\phi})_x = (P\tilde{\phi})_y$$

and hence $P\tilde{\phi}$ satisfies (37). Thus we can find a section ψ of E over $\mathcal{G}^{(0)}$ such that $P\tilde{\phi} = \tilde{\psi}$. Note that $P_x\tilde{\phi}_x$ is defined since P_x is properly supported.

The uniqueness of the section ψ follows from the fact that the map $\phi \to \tilde{\phi}$ is one-to-one, and the smoothness of ψ follows from Lemma 2. □

The representation given by the following theorem reduces to the trivial representation in the case of a group (see also comments bellow).

Theorem 9. *There exists a canonical representation π_0 of the algebra $\Psi^\infty(\mathcal{G}; E)$ on $\mathcal{C}^\infty(\mathcal{G}^{(0)}, E)$ given by $\pi_0(P)\phi = \psi$, where, using the notation of the previous lemma, ψ is the unique section satisfying $\tilde{\psi} = P\tilde{\phi}$. Moreover $\pi_0(P)$ maps compactly supported sections to compactly supported sections.*

Proof. The fact that π_0 is well defined follows from the uniqueness part of the previous lemma. It is clearly a representation. We only need to check that $\pi_0(P)$ maps compactly supported sections to compactly supported sections. Let $L_1 \subset \mathcal{G}^{(0)}$ be the support of ϕ, $L_2 = \operatorname{supp}(P)$. Then the support of $\pi_0(P)$ is contained in L_2L_1. □

Assume that E is a trivial line bundle. $\mathcal{C}^\infty(M)$ and $\Gamma(A)$ act naturally on $\mathcal{C}^\infty(M)$ and this action satisfies the relations (22), which means that it gives rise to a representation of $U(A) = \mathrm{Diff}(\mathcal{G})$ on $\mathcal{C}^\infty(M)$. Then π_0 is an extension of this representation. If $\mathcal{G} = G$ is a group, then π_0 extends the trivial representation. In order to generalize this fact to arbitrary representations of G we need the following definition.

Definition 10. An equivariant bundle (V, ρ) on $\mathcal{G}^{(0)}$ is a differentiable vector bundle E together with a bundle isomorphism $\rho : d^*(V) \longrightarrow r^*(V)$ satisfying $\rho(gh) = \rho(g)\rho(h)$.

An equivariant bundle is also called a representation of $\mathcal{G}$. Given an equivariant bundle (V, ρ), we can define a representation π_ρ of the groupoid algebra $\mathcal{C}_\mathrm{c}^\infty(\mathcal{G}, d^*(\mathcal{D}))$ on $\mathcal{C}_\mathrm{c}^\infty(\mathcal{G}^{(0)}, V)$ by the formula

$$(\pi_\rho(f)\phi)(x) = \int_{\mathcal{G}_x} f(h^{-1})\rho(h^{-1})\phi(r(h)). \tag{38}$$

Note that the integration is defined and gives an element of V_x since $f(h^{-1})\phi(r(h))$ is in $\mathcal{C}_\mathrm{c}^\infty(r^*(V) \otimes \Omega_d)$ and hence that $f(h^{-1})\rho(h^{-1})\phi(r(h))$ is a smooth compactly supported section of $d^*(V) \otimes \Omega_d$.

The following proposition has no obvious analog in the classical theory because the pair groupoid has no nontrivial representations. If one moves one step up and considers the fundamental groupoid, nontrivial representations exist, and the following lemma says that geometric operators (i.e. the ones that lift to the universal covering space) act on sections of flat bundles. A representation of a groupoid thus resembles a flat bundle.

Proposition 7. *Let (V, ρ) be an equivariant bundle and E an arbitrary bundle on $\mathcal{G}^{(0)}$. There exists a natural morphism $T_\rho : \Psi^\infty(\mathcal{G}; E) \to \Psi^\infty(\mathcal{G}; V \otimes E)$ and hence there exist canonical actions $\pi_\rho = \pi_0 \circ T_\rho$ of $\Psi^\infty(\mathcal{G})$ on $\mathcal{C}^\infty(\mathcal{G}^{(0)}, E \otimes V)$ and $\mathcal{C}^\infty_{\mathrm{c}}(\mathcal{G}^{(0)}, E \otimes V)$, which extend the representation defined in (38).*

Proof. Let

$$W_{\rho,x} : \mathcal{C}^\infty_{\mathrm{c}}(\mathcal{G}_x; r^*(E)) \otimes V_x = \mathcal{C}^\infty_{\mathrm{c}}(\mathcal{G}_x; r^*(E) \otimes d^*(V)) \to \mathcal{C}^\infty_{\mathrm{c}}(\mathcal{G}_x; r^*(E \otimes V))$$

be the isomorphism defined by ρ as in Definition 10. It is easy to see that this gives an isomorphism $W_\rho : \mathcal{C}^\infty_{\mathrm{c}}(\mathcal{G}; r^*(E) \otimes d^*(V)) \to \mathcal{C}^\infty_{\mathrm{c}}(\mathcal{G}; r^*(E \otimes V))$. Define an operator on $\mathcal{C}^\infty_{\mathrm{c}}(\mathcal{G}_x; r^*(E \otimes V))$ by the formula

$$(T_\rho(P))_x = W_{\rho,x}(P_x \otimes id_{V_x}) W^{-1}_{\rho,x}.$$

The relation $W_{\rho,x}(U_g \otimes \rho(g)) = U_g W_{\rho,x}$ shows that the family $(T_\rho(P))_x$, $x \in \mathcal{G}^{(0)}$ satisfies the invariance condition $(T_\rho(P))_x U_g = U_g (T_\rho(P))_y$, for $d(g) = x$ and $r(g) = y$. The uniform support condition is satisfied since $\mathrm{supp}(T_\rho(P)) = \mathrm{supp}_\mu(P)$. It follows that the family $(T_\rho(P))_x$ defines an operator $T_\rho(P)$ in $\Psi^\infty(\mathcal{G}; V \otimes E)$. The multiplicativity condition $T_\rho(PQ) = T_\rho(P)T_\rho(Q)$ follows from definition and hence T_ρ is a morphism. □

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