COURSE 1

SPECTRAL FLUCTUATIONS IN DISORDERED METALS

G. MONTAMBAUX

Laboratoire de Physique des Solides, Associé au CNRS
Université Paris–Sud
91405 – Orsay
France

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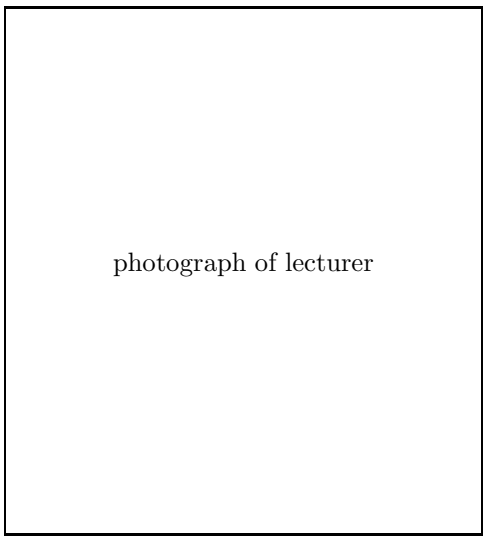
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Contents

1. Introduction, the relevant scales

With the development of the nanotechnologies, new possibilities recently emerged to find electronic properties which reflect the discrete aspect of energy spectra. The objects we shall describe in this course are metals which will be considered as quantum mechanical objects. Long time ago Kubo realized that in a small system of typical size L, the inter-level spacing $\Delta \propto L^{-d}$ may become as large as the temperature. The discreteness of the spectrum should manifest itself through a qualitative change in thermodynamic properties[1, 2]. It turns out that beside Δ, another characteristic energy scale E_c emerges (*the Thouless energy*), which determines the thermodynamic properties in a crucial way.

It is the purpose of this course to review the properties of the metallic spectra and relate them to physical quantities, transport and thermodynamics. In a first approximation, a metal can be considered as a complex quantum system which shares universal spectral properties with other so-called chaotic systems, like nuclei, molecules, models of billiards. One essential property is that the energy levels are strongly correlated and present the phenomenon of *spectral rigidity* [2].

Such a system is described by four length scales: the sample size L, the mean free path l_e which describes the elastic collisions, the Fermi wave length λ_F which depends on the density of electrons and the coherence length L_φ. This last scale is very important because the effects we aim to describe in this course result from the phase coherence of the wave functions and thus disappear beyond L_φ[3]. Smaller distances define *the mesoscopic regime*. In this course, we shall mostly consider an electron gas in the following limits:

$$\lambda_F \ll l_e \ll L \ll L_\varphi \tag{1.1}$$

which correspond to a *weakly disordered* ($\lambda_F \ll l_e$), *mesoscopic* ($L \ll L_\varphi$) metal in the *diffusive regime* ($l_e \ll L$). When the disorder becomes so large that l_e is reduced to a length of order λ_F, the wave functions become

localized on a typical scale ξ called the localization length[5, 6]. Here we shall study only the diffusive regime where $\xi \to \infty$.

In this regime, an electron moves diffusively because it experiences many elastic collisions while moving in the sample. The typical distance covered by the diffusive particle in a time t varies as $r^2(t) \simeq Dt$. The diffusive motion is thus characterized by a new time scale τ_D which is the typical time for an electron to travel through the sample. It is defined as $\tau_D = L^2/D$. To this time scale corresponds a new energy scale E_c called the Thouless energy:

$$E_c = \frac{\hbar}{\tau_D} = \frac{\hbar D}{L^2} \tag{1.2}$$

where the classical diffusion coefficient D is given by

$$D = \frac{v_F^2 \tau_e}{d} = \frac{v_F l_e}{d} \tag{1.3}$$

$\tau_e = l_e/v_F$ is the elastic collision time. v_F is the Fermi velocity. It is interesting to notice that the d.c. residual ($T = 0K$) conductance σ can be directly related to the Thouless energy. Using the Einstein relation $\sigma = e^2 \rho_0 D$, where ρ_0 is the average density of states, and the Ohm relation between the conductance G and the conductivity, $G = \sigma L^{d-2}$, the dimensionless conductance g can be written as

$$g = \frac{G}{e^2/\hbar} = \frac{\sigma L^{d-2}}{e^2/\hbar} = \frac{E_c}{\Delta} \equiv N(E_c) \tag{1.4}$$

since the average interlevel distance Δ is $1/(\rho_0 L^d)$. $N(E_c)$ is thus the average number of states in a strip of the spectrum of width E_c. We shall see that, beside Δ, the energy scale E_c enters in an essential way to describe the spectral properties of mesoscopic systems. Although disordered systems

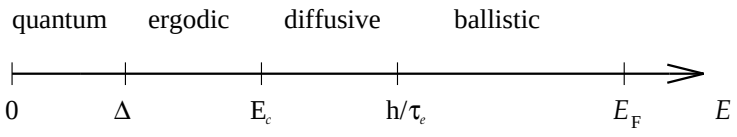

Fig. 1. Relevant energy scales

have been mainly studied for their transport properties, there has been in the recent years an increasing interest in their spectral properties. This is in large part because of the development of nanotechnologies which allows

the fabrication of mesoscopic devices and the detection of small signals. In particular, the discovery of persistent currents in mesocopic isolated rings has motivated the study of the statistical properties of energy levels in the presence of an external parameter, a Aharonov-Bohm (AB) magnetic flux, which breaks the Time-Reversal Symmetry (TRS)[7, 8, 9, 10]. This breakdown of TRS has the effect of changing the spectral properties. It has become also possible to measure the spectral correlations by transport experiments in which it is possible to do a spectroscopy of the levels[11].

Another motivation for the study of spectral correlations comes from an argument due to Thouless which relates the conductance to the typical variation of the energy levels in a change of the boundary conditions[12, 13]. This argument links the transport properties to the spectral properties and will be discussed in details. Among the signatures of the phase coherence on transport properties, a very important one is the property of Universal Conductance Fluctuations. In the mesoscopic regime , the conductance is not a self-averaging quantity and its distribution is characterized by a universal variance[14, 15, 16, 17]

$$\langle G^2\rangle - \langle G\rangle^2 \sim (\frac{e^2}{\hbar})^2 \tag{1.5}$$

We shall see that this universal variance is closely related to the *spectral rigidity* mentioned above[17].

The course is organized as follows: the next section presents the microscopic model used for disordered metals. It describes *non-interacting particles* moving in a random potential. It is shown that low energy spectral properties exhibit the universal features of the Random Matrix Theory (RMT) and that deviations exist above the energy scale E_c. In section 3, a semi-classical picture relates the spectral correlations to the description of a diffusive particle in a random medium. Section 4 is devoted to the persistent current in a mesoscopic ring and its relation with the fluctuations of the local or global density of states. The curvature distribution and its relation with the conductance are analyzed in section 5. Then it is seen that the parametric correlations, namely the correlations between energy levels at different values of an external parameter like an AB flux, exhibit a universal behavior after rescaling of the energy and flux scales (section 6).

This course overviews only some aspects of the spectral fluctuations in disordered metals, especially those using the semi-classical approach[18]. The subject has been covered by many aspects on recent review papers[4, 19, 20, 21]. Throughout this course, unless otherwise specified, we shall set $\hbar = 1, L^d = 1$, $\Delta = 1$ and we shall describe spinless electrons.

2. Metal as a quantum chaotic system

The study of spectral statistics in complicated systems has been initiated by Wigner, Dyson, Mehta and others to describe the spectra of nuclei [22, 23, 24, 25, 26]. These authors describe the statistical properties of matrix Hamiltonians, with a Gaussian distribution of the elements around a zero average. A remarkable feature of the Random Matrix Theory (RMT) is that the distribution of levels depends only on the symmetry of the Hamiltonian. In particular, if the Hamiltonian is invariant under time–reversal symmetry, the statistical ensemble of matrices is invariant under orthogonal transformations and is called Gaussian Orthogonal Ensemble (GOE). If the system is not invariant under time–reversal symmetry, for example in the presence of a magnetic field, the statistical ensemble of matrices is invariant under unitary transformations and is called Gaussian Unitary Ensemble (GUE). The RMT has been applied to a variety of very different physical situations in nuclear, atomic and molecular physics [2, 22, 23, 24, 25, 26]. The remarkable outcome is that it gives a universal description of complex spectra and we shall use it to describe the spectra of disordered metals.

The relevance of the RMT for a metal has been first pointed out by Gorkov and Eliashberg, in their study of ac response of small metallic particles[27]. In their work, the complexity was not coming from the many–body nature of the Hamiltonian like in nuclear physics but from the scattering of electrons on the random shape of the boundaries, the system having the structure of a billiard with rough boundaries. Here we describe a disordered metal where the electrons are scattered elastically by fixed impurities.

Neglecting the electron-electron interactions, a disordered metal is usually described by a one particle Schrödinger equation

$$\mathcal{H}\psi = \frac{-\hbar^2}{2m}(\boldsymbol{\nabla} - i\frac{e\boldsymbol{A}}{\hbar c})^2\psi + V(\mathbf{r})\psi = E\psi \tag{2.6}$$

$\boldsymbol{A}$ is the vector potential. The disorder potential is commonly modeled as:

$$\langle V(\mathbf{r})\rangle = 0 \qquad\qquad \langle V(\mathbf{r})V(\mathbf{r}')\rangle = \frac{\delta(\mathbf{r}-\mathbf{r}')}{2\pi\rho_0\tau_e} \tag{2.7}$$

τ_e is the elastic collision time. This structure is convenient for analytical calculations and the essential physics does not depend on the detailed choice of the disorder. The discrete version of this Hamiltonian, called the Anderson tight-binding Hamiltonian[5], is commonly used for numerical

calculations

$$\mathcal{H}\psi_i = -\sum_j t_{ij} \exp(i\frac{e}{\hbar c}\int_i^j Adl)\psi_j + V(i)\psi_i \tag{2.8}$$

where the wave function ψ_i is defined on the sites i of a lattice. In the simplest version of this model, the t_{ij} are taken as constant $t_{ij} = t$ and $V(i)$ is a random variable with a uniform distribution in the range $[-W/2, W/2]$. This Hamiltonian, first described by Anderson, exhibits localization properties and a Metal-Insulator transition in 3D [5, 6]. Fig. 2 shows a typical spectrum in the diffusive and localized regimes, in the presence of an external AB flux (see sect. 4). The goal of this course is to describe the statistical properties of such spectra. Let us first recall the usual quantities used to measure the level fluctuations[24, 25, 26] (The energies are measured in units of the mean level spacing Δ:

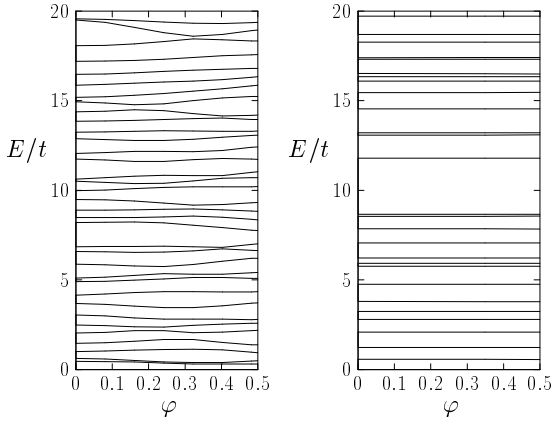

Fig. 2. A typical spectrum of a metal in the diffusive regime (left), and in the localized regime (right). $\varphi = \phi\phi_0$ is the normalized Aharonov-Bohm flux, see sect. 4.

– The distribution $P(s)$ of spacing s between consecutive levels. In the RMT, it is well described by the Wigner-surmise:

$$P(s) \propto s^\beta \exp(-c_\beta s^2) \tag{2.9}$$

where β depends on the symmetry of the Hamiltonian. When there is no correlation between levels, it has a Poisson behavior:

$$P(s) = \exp(-s) \tag{2.10}$$

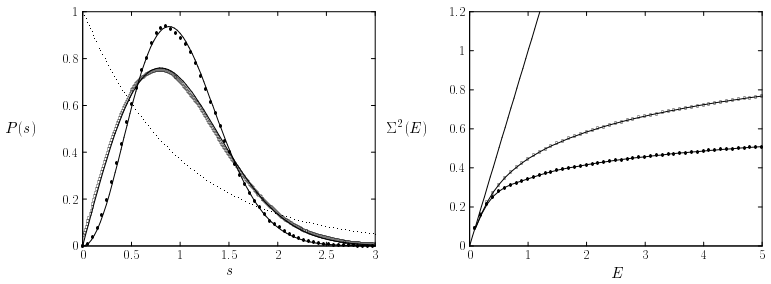

Fig. 3. $P(s)$ and $\Sigma^2(E)$ for a metal in the diffusive regime, with (black dots) and without (open dots) magnetic flux[30]. They are very well described by the Random Matrix Theory.

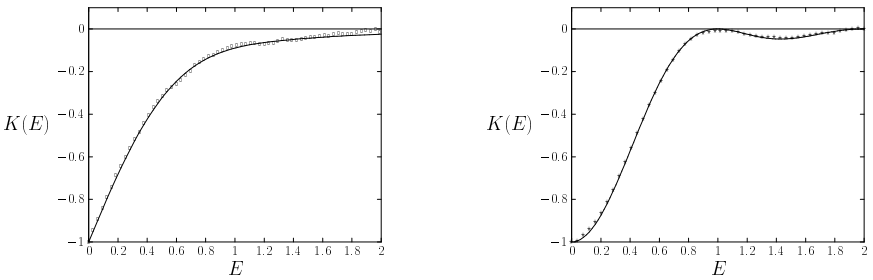

Fig. 4. Two-point correlation function for a metal in the diffusive regime, with (left) and without (right) magnetic flux[31]. The continuous line is the RMT prediction A δ function at the origin is not shown.

– The two-point correlation function of the Density of States (DOS):

$$K(\epsilon_1, \epsilon_2) = \langle \rho(\epsilon_1)\rho(\epsilon_2)\rangle - \rho_0^2 \tag{2.11}$$

The average $\langle ...\rangle$ is done on energy. Here the average will be also made on different disorder realizations. In the diffusive regime, these two procedures are equivalent. If, after averaging, the DOS is constant, the function $K(\epsilon_1, \epsilon_2)$ depends only on the difference $\epsilon_1 - \epsilon_2$. In the RMT and for large separation ϵ, it varies as $K(\epsilon) \to -1/(\beta\pi^2\epsilon^2)$. At $\epsilon = 0$, it has a δ peak which describes the self-correlation of the levels. The full expression is simple in the GUE case: $K(\epsilon) = \delta(\epsilon) - \sin^2(\pi\epsilon)/(\pi\epsilon)^2$ (see fig. 4).

– The number variance $\Sigma^2(E)$

$$\Sigma^2(E) \equiv \langle \delta N^2(E)\rangle = \langle N^2(E)\rangle - \langle N(E)\rangle^2 \tag{2.12}$$

measures the fluctuation of the number of levels $N(E)$ in a strip of width E. By definition, the number variance can be written in terms of this

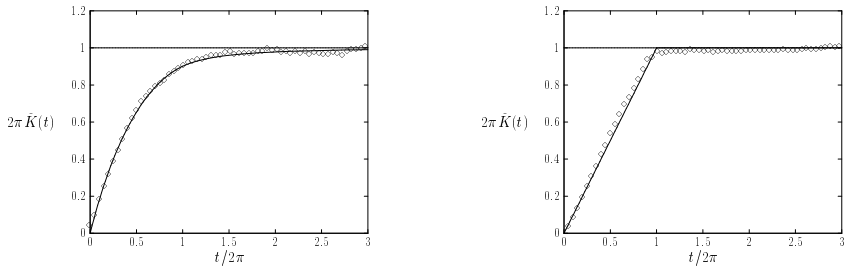

Fig. 5. The structure factor $\tilde{K}(t)$ for a metal in the diffusive regime, with and without magnetic flux.

two-point correlation function.

$$\Sigma^2(E) = \int_0^E \int_0^E K(\epsilon_1 - \epsilon_2) d\epsilon_1 d\epsilon_2 = 2\int_0^E (E-\epsilon) K(\epsilon) d\epsilon \tag{2.13}$$

In the RMT , for $E > \Delta$, it has a logarithmic behavior:

$$\Sigma^2(E) \simeq \frac{2}{\pi^2\beta} \ln(E/\Delta) + Cte \tag{2.14}$$

Although the number variance is very frequently used in the literature, this quantity is not always the most appropriate to describe the correlation because it is a double integral of the DOS-DOS correlation function $K(\epsilon)$. Thus the behavior of this quantity at an energy scale E depends on the behavior of the two-point correlation function for *all* energies smaller than E.

– The Fourier transform $\tilde{K}(t)$ is called the spectral form factor

$$\tilde{K}(t) = \frac{1}{2\pi} \int K(\epsilon) \exp(i\epsilon t) d\epsilon \tag{2.15}$$

The advantage of this quantity is to be directly related to the diffusive motion of a classical particle (sect. 3).

At small times $t \ll \tau_H$ where $\tau_H = 2\pi\hbar/\Delta$ is called the Heisenberg time, the form factor varies linearly with time $\tilde{K}(t) \to t/(2\pi^2\beta)$ and it saturates to a constant value $1/2\pi$ for $t \to \infty$. This constant is simply the Fourier transform of the δ peak which describes the self-correlation of the levels[28]. Figures 3,4,5 present numerical calculations of the different correlation functions, with the Anderson Hamiltonian[29, 30, 31]. They are very well fitted by the RMT. Using the supersymmetric technique to calculate $K(\epsilon)$ in the microscopic model (eq. 2.6), Efetov has shown that its expression actually coincides with the RMT result[32].

However, fig. 6 shows that above a given energy which is size and disorder dependent and which has been identified with E_c, the number variance exhibits deviations and increases rapidly with energy. These deviations

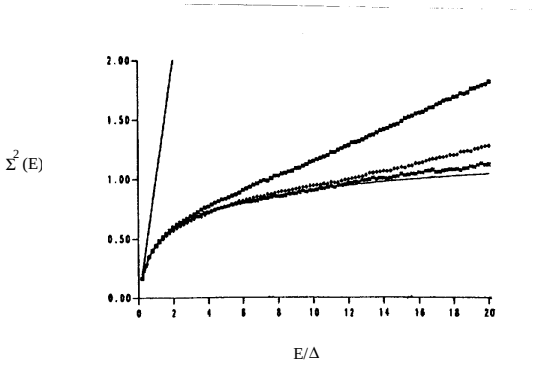

Fig. 6. The number variance exhibits deviations to RMT above E_c

from RMT are not surprising: the Anderson Hamiltonian has a very different structure from a random matrix. It has many zero matrix elements. As we have seen in the introduction, there are other energy scales not included in the RMT. Thus we expect deviations from the universal behavior. RMT systems are "ergodic" in the sense that their eigenfunctions cover uniformly the phase space and have no internal structure. This is clearly not the case in a metal at small times or large energies where a diffusive particle cannot explore the entire space (fig. 7). More precisely, for small times $t_E \ll \tau_D$, i.e. large energies $E \gg E_c$, the spatial correlations extend on a scale $L_E = \sqrt{Dt_E} = \sqrt{\hbar D/E}$, so that the system consists of $(L/L_E)^d$ independent pieces, fig. 7. Consequently, $\Sigma^2(E) \sim (L/L_E)^d \sim (E/E_c)^{d/2}$ [17].

In the next chapter, we describe the specific properties of the metallic

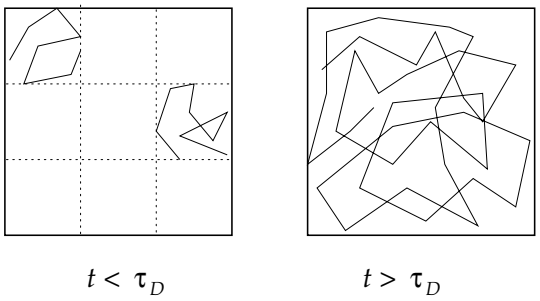

Fig. 7. Schematic diffusion at small times (*diffusive regime*) and large times (*ergodic regime*)

spectra and their deviations from the universal properties given by the RMT.

3. Level correlations in a disordered metal

We assume that the disorder is weak, so that the energy scales obey the hierarchy of eq. 1.1. Thus the dimensionless conductance $g \gg 1$. In this regime, we want now to account for level correlations within and beyond the RMT regime.

First it is useful to introduce the following important operator known as the Green function or resolvent:

$$G^{R,A}(E) = \frac{1}{E - \mathcal{H} \pm i0} \tag{3.16}$$

In the site representation, the elements of the Green function are given by

$$G^{R,A}(\mathbf{r}, \mathbf{r}', E) = \langle \mathbf{r}|G^{R,A}(E)|\mathbf{r}'\rangle = \sum_n \frac{\phi_n^*(\mathbf{r})\phi_n(\mathbf{r}')}{E - E_n \pm i0} \tag{3.17}$$

E_n and ϕ_n are eigenenergies and eigenstates of the Hamiltonian. The Fourier transform $G^R(\mathbf{r}, \mathbf{r}', t) = \frac{1}{2i\pi}\int G^R(\mathbf{r}, \mathbf{r}', E)e^{-iEt}dE$ has the following interesting property: Take a wave function $\psi(\mathbf{r}, t = 0)$, solution of the time dependent Schrödinger equation (2.6). The time evolution of this wave function can be written as:

$$\psi(\mathbf{r}, t) = \int d\mathbf{r}' G^R(\mathbf{r}, \mathbf{r}', t)\psi(\mathbf{r}', 0) \tag{3.18}$$

So $G^R(\mathbf{r}, \mathbf{r}', t)$ "propagates" a particle from $(\mathbf{r}', 0)$ to $(\mathbf{r}, t)$. It results immediately that the probability for a particle in $\mathbf{r}'$ at $t = 0$, to be in $\mathbf{r}$ at t is

$$|G^R(\mathbf{r}, \mathbf{r}', t)|^2 = G^R(\mathbf{r}, \mathbf{r}', t)G^A(\mathbf{r}', \mathbf{r}, -t) \tag{3.19}$$

If the particle is prepared at the energy ϵ_F, the probability becomes[33, 34]:

$$P(\mathbf{r}, \mathbf{r}', t) = \frac{1}{2\pi}\int P(\mathbf{r}, \mathbf{r}', \omega)e^{-i\omega t}d\omega \tag{3.20}$$

with

$$P(\mathbf{r}, \mathbf{r}', \omega) = \frac{1}{2\pi\rho_0}\langle G^R(\mathbf{r}, \mathbf{r}', \epsilon_F + \frac{\omega}{2})G^A(\mathbf{r}', \mathbf{r}, \epsilon_F - \frac{\omega}{2})\rangle \tag{3.21}$$

The DOS can be expressed in term of the Green function as

$$\rho(E) = -\frac{1}{\pi}\int ImG^R(\mathbf{r},\mathbf{r},E)d\mathbf{r} \tag{3.22}$$

It is also useful to define the *local* DOS as

$$\rho(r,E) = -\frac{1}{\pi}ImG^R(\mathbf{r},\mathbf{r},E) \tag{3.23}$$

The two-point correlation function can thus be written as:

$$K(\epsilon_1,\epsilon_2) = \frac{1}{2\pi^2}\int\int d\mathbf{r}d\mathbf{r}'\langle G^R(\mathbf{r},\mathbf{r},\epsilon_1)G^A(\mathbf{r}',\mathbf{r}',\epsilon_2)\rangle_c \tag{3.24}$$

where $\langle ...\rangle_c$ is the connected part of the average. It is essential to notice that the Green function appearing in the DOS and its correlation function is a *diagonal* quantity (in space representation) which depends on the propagation of particles from some origin to itself. All spectral quantities will thus depend only on $G(\mathbf{r},\mathbf{r},E)$.

Another very important quantity which depends only on the diagonal Green function is the *return* probability to the origin $P(t) = P(\mathbf{r},\mathbf{r},t)$. Its Fourier transform $P(\omega)$ is given by

$$P(\omega) = \frac{V}{2\pi\rho_0}\langle G^R(\mathbf{r},\mathbf{r},\epsilon_F+\frac{\omega}{2})G^A(\mathbf{r},\mathbf{r},\epsilon_F-\frac{\omega}{2})\rangle \tag{3.25}$$

Upon averaging, $P(\omega)$ is independent of the position $\mathbf{r}$.

One sees that $K(\epsilon_1,\epsilon_2)$ and $P(\omega)$ have similar structures. The goal of this section will be to relate these two quantities. To calculate $P(t)$, we assume that the metal is well described by a random potential in which the electrons experience a diffusive motion of classical particles so that $P(\omega)$ is a classical quantity equal to $P_{cl}(\omega)$[33, 35]. The probability to diffuse from $\mathbf{r}'$ to $\mathbf{r}$ is thus given by the solution of the classical diffusion equation i.e.

$$P_{cl}(\mathbf{r},\mathbf{r}',t) = \sum_{\mathbf{q}} e^{-Dq^2t}e^{i\mathbf{q}(\mathbf{r}-\mathbf{r}')} \tag{3.26}$$

where the diffusion modes $\mathbf{q}$ are quantized by the boundary conditions. In the limit of an infinite system, it takes the familiar form

$$P_{cl}(\mathbf{r},\mathbf{r}',t) = \frac{V}{(4\pi Dt)^{d/2}}e^{-|\mathbf{r}-\mathbf{r}'|^2/4Dt} \tag{3.27}$$

where D is the diffusion coefficient. The return probability and its Fourier transform are thus given by[36]:

$$P_{cl}(t) = \sum_{\mathbf{q}} e^{-Dq^2t} \qquad P_{cl}(\omega) = \sum_{\mathbf{q}}\frac{1}{-i\omega+Dq^2} \tag{3.28}$$

In sect. 3.2, we shall show that the two-point correlation function $K(\omega = \epsilon_1 - \epsilon_2)$ is related to the return probability $P(\omega)$.

3.1. A brief reminder about weak-localization

Before describing the structure of the spectral correlations, let us first recall a brief qualitative derivation of the first quantum correction to the classical conductivity, called *weak-localization* correction [33, 35, 38]. Linear response theory shows that the d.c. $T = 0K$ average conductivity[37] has the following structure:

$$\langle\sigma\rangle = -\frac{e^2\hbar^3}{2\pi m^2 V}\int\int d\mathbf{r}d\mathbf{r}'\langle\partial_x G^R(\mathbf{r},\mathbf{r}',\epsilon_F)\partial_{x'}G^A(\mathbf{r}',\mathbf{r},\epsilon_F)\rangle \tag{3.29}$$

By writing the Green function as a sum of contributions from classical paths, as it will be explained in more details in the next subsection (eq. 3.33), the conductivity has the following structure:

$$\langle\sigma\rangle \propto \sum_{j,k}\langle B_j B_k^* e^{i(S_j - S_k)/\hbar}\rangle \tag{3.30}$$

where B_j and S_j are the amplitude and the action associated to each path j. For most of the pairs of trajectories, $S_j - S_k > 2\pi\hbar$, so that their contribution to the conductivity cancels in average. The classical conductivity is given by the sum of the intensities: $\sigma_{cl} \propto \sum_j |B_j|^2$. However, there is a class of trajectories which can also contribute to the conductivity: those which form *closed loops*. Such a loop can be traveled in clockwise

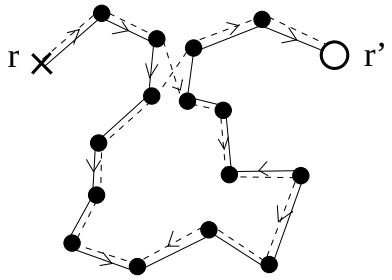

Fig. 8. The two closed trajectories which make a closed loop are time-reversal symmetric. They give an additional contribution to the conductivity called weak-localization correction. This correction is proportional to the number of such closed trajectories.

or anti-clockwise directions. Both trajectories, j and its time-reversed j^T,

have the same action, so that they interfere constructively. As a result, in addition to the classical contribution, there is a correction of the form $\Delta\sigma \propto \sum'_j B_j B^*_{j^T}$ where the sum extends over the closed trajectories. The sign of the correction is negative because the trajectories j and j^T have opposite momenta. As a result there is an enhanced probability to scatter in backwards direction. The conductivity is thus *reduced*. This effect is often called *coherent backscattering or weak-localization*. This is a phase coherent effect because only trajectories of size smaller than L_φ will contribute to the correction. The amplitude of the correction is proportional to the total number of loops of any size. The number of loops of length $v_F t$ being proportional to the return probability $P_{cl}(t)$, one deduces that the total correction is given by:

$$\frac{\Delta\sigma}{\sigma} \propto -\frac{\lambda_F^{d-1} v_F}{V} \int_{\tau_e}^{\tau_\varphi} P_{cl}(t) dt \tag{3.31}$$

λ_F^{d-1} is the transverse area associated to a semi-classical trajectory. The integral is calculated between τ_e, the smallest time for diffusion, and τ_φ, the time after which the electron looses phase coherence. τ_φ is given by $L_\varphi^2 = D\tau_\varphi$. A magnetic field[39] or a Aharonov-Bohm flux[40], by breaking the time-reversal symmetry, destroys the weak-localization correction[41].

In the next subsection, we calculate the spectral function $K(\omega)$, by using very similar arguments.

3.2. Semi-classical description of energy levels correlations

We survey here a method which gives a very physical description of the level correlations because it directly relates them to the structure of the classical motion of a diffusive particle. More precisely, one can get a relation between the spectral form factor $\tilde{K}(t)$ and the classical return probability for a particle to return to the origin $P(t)$. This description has been developed for diffusive electrons by Argaman, Imry and Smilansky[18] from ideas originally applied to other various classically chaotic systems[42, 43, 44]. We present here a brief and slightly different derivation. The first step is to write the Green function as a path integral [45, 46]

$$G^R(\mathbf{r}, \mathbf{r}', t) = \int_{\mathbf{r}'}^{\mathbf{r}} \mathcal{D}[\mathbf{x}] \exp(\frac{i}{\hbar} S[\mathbf{x}]) \,, \tag{3.32}$$

where $S[\mathbf{x}] = \int \mathbf{p} d\mathbf{x}$ is the classical action for the path $\mathbf{x}(t)$ going from $\mathbf{x}(0) = \mathbf{r}'$ to $\mathbf{x}(t) = \mathbf{r}$. The integral extends over all paths going from $\mathbf{x}(0) = \mathbf{r}'$ to $\mathbf{x}(t) = \mathbf{r}$. The classical paths given by $\delta S[\mathbf{x}]/\delta\mathbf{x} = 0$ give the

main contribution to the integral (3.32) so that the Green function can be written as

$$G^R(\mathbf{r},\mathbf{r}',E) = \sum_j A_j(\mathbf{r},\mathbf{r}',E) e^{iS_j(\mathbf{r},\mathbf{r}',E)/\hbar}\,, \tag{3.33}$$

This expression results from a stationary phase approximation near classical paths[47, 48]. The amplitudes A_j result from a Gaussian integration around the classical trajectories. Using the definition (3.25) of the return probability $P(\omega)$, one gets

$$P(\omega) = \frac{1}{2\pi\rho_0}\langle \sum_{j,k} A_j(\mathbf{r}) A_k^*(\mathbf{r}) e^{i[S_j(E+\frac{\omega}{2})-S_k(E-\frac{\omega}{2})]/\hbar}\rangle\,, \tag{3.34}$$

This sum is done over *closed* paths. We have defined $A_j(\mathbf{r}) \equiv A_j(\mathbf{r},\mathbf{r})$. Moreover, for a closed path, the action is independent of the starting point. The main energy dependence is contained in the phase factors and we have suppressed it in the amplitudes A_j. It has been argued by Berry that the phase factors with large actions such that $S_j - S_k \gg 2\pi\hbar$ should cancel on average[28, 49]. As a result, in the absence of time reversal symmetry, only the diagonal terms are kept in the sum. Using the relation $T_j = dS_j/dE$ between the period and the energy dependence of the action for a closed path, the phase factors can be expanded as $S_j(E+\omega/2) = S_j(E) + \omega T_j/2$, so that one obtains:

$$P(\omega) = \frac{1}{2\pi\rho_0}\langle \sum_j |A_j(\mathbf{r})|^2 e^{i\omega T_j}\rangle \tag{3.35}$$

By Fourier transformation, the return probability can be written as:

$$P(t) = \frac{1}{2\pi\rho_0}\langle \sum_j |A_j(\mathbf{r})|^2 \delta(t-T_j)\rangle \tag{3.36}$$

After disorder averaging, this quantity is independent of the origin $\mathbf{r}$. We now turn to the calculation of the two-point correlation function, that we write in a symmetrized form. Using the expression in terms of the Green functions, doing the same manipulations as for the return probability, one obtains

$$\begin{aligned}
\tilde{K}(E,t) &= \frac{1}{2\pi}\int \langle \rho(E+\frac{\epsilon}{2})\rho(E-\frac{\epsilon}{2})\rangle e^{i\epsilon t} d\epsilon \\
&= \frac{1}{2\pi^2}\langle \sum_j \int\int d\mathbf{r} d\mathbf{r}' A_j(\mathbf{r}) A_j^*(\mathbf{r}')\delta(t-T_j)\rangle \\
&= \frac{1}{2\pi^2}\langle \sum_j |A_j|^2 \delta(t-T_j)\rangle\,,
\end{aligned} \tag{3.37}$$

where $A_j \equiv \int A_j(\mathbf{r}) d\mathbf{r}$. Clearly, the expressions for $P(t)$ and $\tilde{K}(t)$ are very similar. To relate them, it is now important to make the following distinction. The semi-classical motion is characterized by an ensemble of closed *trajectories*. $P_j = |A_j(\mathbf{r})|^2$ is the probability to be on the trajectory j with origin $\mathbf{r}$. Many other trajectories follow the same path but they start from different points $\mathbf{r}$ and $\mathbf{r}'$. This ensemble of trajectories, who have the same action, is called an *orbit*. The probability to be on a given orbit j is

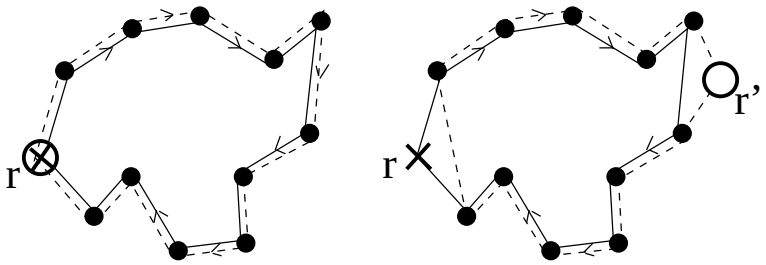

Fig. 9. a) Two identical trajectories starting from the same origin and belonging to the same orbit: these trajectories contribute to $P(t)$. b) Two trajectories with different origin belonging to the same orbit: these trajectories contribute to $\tilde{K}(t)$.

$|A_j|^2 = \int\int d\mathbf{r} d\mathbf{r}' A_j(\mathbf{r}) A_j^*(\mathbf{r}')$. One sees that for one given $\mathbf{r}$, the possible choice for $\mathbf{r}'$ on the orbit j is proportional to the volume of this orbit. This "semi-classical" volume is $\lambda_F^{d-1} v_F T_j$ were λ_F^{d-1} is the section of the classical orbit and $v_F T_j$ is its length. Recalling that $\lambda_F^{d-1} v_F$ is proportional to the inverse DOS, one deduces (this simple argument cannot reproduce here the correct prefactor):

$$|A_j|^2 = \frac{1}{4\pi\rho_0} P_j T_j \tag{3.38}$$

This result has been obtained for the first time by Hannay and Ozorio de Almeida in the ergodic case ($P_j = 1$) [50, 49]. After summation over the orbits, comparing eqs. 3.36 and 3.37, one obtains:

$$\tilde{K}(t) = \frac{1}{4\pi^2} t P_{cl}(t) \tag{3.39}$$

This semi-classical expression is believed to be valid for small orbits , with period t smaller than the quantum (or Heisenberg) time $\tau_H = 2\pi\hbar/\Delta$[18, 49]. Fourier transform of eq. 3.39 gives

$$K(\omega) = \frac{1}{2\pi^2} \mathrm{Im} \frac{\partial P_{cl}(\omega)}{\partial \omega} = -\frac{1}{2\pi^2} \mathrm{Re} \sum_q \frac{1}{(-i\omega + Dq^2)^2} \tag{3.40}$$

and the number variance is obtained from double integration (eq. 2.12):

$$\Sigma^2(E) = 8\int_0^\infty dt \frac{\tilde{K}(t)}{t^2}\sin^2(\frac{Et}{2}) = \frac{2}{\pi^2}\int_0^\infty dt \frac{P(t)}{t}\sin^2(\frac{Et}{2}) \qquad (3.41)$$

In the above diagonal approximation, eq. 3.35, we kept the terms where $k = j$. However, in the case where the system is time-reversal invariant, for each orbit j there is an orbit which is reversed by time inversion $k = j^T$ and which have the same action. Thus, the non-diagonal terms $k = j^T$ also survive on average. This *interference* term is not taken into account in the classical diffusion so that the semi-classical probability to return to the origin is actually *twice* the classical probability $P(t) = 2P_{cl}(t)$. This description is called *semi-classical* because it is essentially a classical picture in which the quantum mechanics only appears through a phase coherent classical contribution. The two contributions are called in the diagrammatic language the diffuson (diagonal) and the cooperon (interference) contributions. Eq. 3.40 has been obtained for the first time directly by Altshuler and Shklovskii, using a diagrammatic calculation[17].

In order to take into account the loss of coherence due to dephasing events, an exponential damping $e^{-\gamma t}$ has to be added in the interference term of $P(t)$ which cuts the contribution of long orbits, so that the diffusion pole in 3.40 has a gap $-i\omega + Dq^2 \to -i\omega + Dq^2 + \gamma$. The inverse scattering time γ is related to the coherence length L_φ: $\gamma = \hbar D/L_\varphi^2$ Moreover, the semi-classical approximation breaks down for small energy scales $\omega < \Delta$ so that, even without damping, the divergence in the diffusion pole has to be regularized by the transformation $\omega \to \omega + i\gamma_\Delta$ where γ_Δ is an energy scale of the order of Δ[30]. The correlation function exhibits clearly two distinct regimes:

When $\omega < E_c$ or $\tau > \tau_D$, the diffusion is uniform in the sample (fig. 7): $P(t) = 2/\beta$ so that $\tilde{K}(t)$ varies linearly in time $\tilde{K}(t) = t/(2\beta\pi^2)$. By Fourier transform, one has $K(\omega) = -1/\beta\pi^2(\omega + i\gamma_\Delta)^2$ and the number variance varies logarithmically: $\Sigma^2 = (2/\beta\pi^2)\ln(E/\gamma_\Delta)$. This is the *ergodic* regime well described by the RMT. It is obtained by taking only the $\mathbf{q} = 0$ contribution in the sum 3.40. This contribution is called the *uniform* or *zero mode*. It should be emphasized that this semi-classical approximation cannot describe properly time scales close to τ_H or energy scales $\omega \leqslant \Delta$.

Keeping only the diagonal terms corresponds to the GUE case ($\beta = 2$) and the interference terms double the contribution to the correlation function ($\beta = 1$). This explains why the fluctuations in the GOE case are about twice as large as in the unitary case.

In the opposite limit (fig. 7), when $\omega \gg E_c$ or $\tau \ll \tau_D$, the return probability depends on the space dimensionality d : $P(t) \propto V/(Dt)^{d/2}$ so

that $\tilde{K}(t) \propto t^{1-d/2}/D^{d/2}$ and, for $\omega \gg E_c$:

$$K(\omega) \propto -\frac{1}{\beta\omega^2} \left(\frac{\omega}{g}\right)^{d/2} \cos(\frac{\pi d}{4}) \tag{3.42}$$

This leads to a power law dependence of the number variance[17]

$$\Sigma^2(E) \propto \frac{1}{\beta} \left(\frac{E}{E_c}\right)^{d/2} \tag{3.43}$$

It is remarkable that the sign of the correlation function is now dimension dependent. In three dimensions, $K(\omega)$ varies at large energies positively like $+1/\sqrt{\omega}$ instead of $-1/\omega^2$ for RMT, meaning that the levels tend to *attract* each other at large energies[51]. The power law regime of the variance has been found numerically recently. However, the deviation of the correlation function from RMT is quite hard to observe numerically in the metallic regime[52, 53, 31] when $g \gg 1$.

Finally, when $E > \hbar/\tau_e$, i.e. $t < \tau_e$, the motion is not diffusive anymore and becomes ballistic. In this regime, the spectral rigidity is very weakly dependent on E[54].

3.3. Breakdown of time-reversal symmetry, Aharonov-Bohm flux

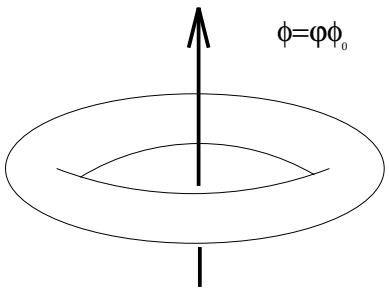

Fig. 10. A ring pierced by a Aharonov-Bohm flux

The above section has stressed the importance of the interference effects in the origin of spectral rigidity. They are affected by the presence of an external parameter which breaks the time-reversal symmetry, for example a Aharonov-Bohm (AB) magnetic flux: one considers a disordered metal having the geometry of a quasi-1D ring of perimeter L pierced by an AB flux ϕ, fig. 10. It is assumed that the vector potential A is a constant ϕ/L

in the ring. The energy spectrum is solution of the Schrödinger equation (2.6) with the periodic boundary condition $\psi(x+L) = \psi(x)$ where x is the coordinate along the ring. A gauge transformation $\psi'(x) = \psi(x)e^{ie\int Adl/\hbar}$ removes the flux from the Hamiltonian: $\mathcal{H}(\phi)\psi = \mathcal{H}(0)\psi'$ but the new wave function ψ' obeys a new boundary condition $\psi'(x+L) = \psi'(x)e^{2i\pi\phi/\phi_0}$, where $\phi_0 = h/e$ is the flux quantum. *An external Aharonov-Bohm flux is thus mathematically equivalent to a change in the boundary condition*[55]. Moreover, it breaks the time reversal symmetry and must lead to a change in the spectral statistics, from the orthogonal to the unitary symmetry.

The transition between the two symmetries for random matrices has been considered by Pandey and Mehta [56, 57]. They have used an interpolating ensemble of matrices of the form $H = H(S) + i\alpha H(A)$ where $H(S)$ and $H(A)$ are real symmetric and antisymmetric matrices of dimension N and variance v^2. $(\alpha = 0)$ is the orthogonal case and $(\alpha = 1)$ describes the unitary symmetry. These authors have shown that the transition between GOE and GUE is driven by the single parameter $\Lambda = v^2\alpha^2/\Delta^2$. For Gaussian matrices, the average interlevel spacing is not a constant in the spectrum so that the transition depends on the position in the spectrum[56, 57]. For a Gaussian matrix of size N and variance v^2, the mean level spacing in the band center is given by $\Delta = \pi v/\sqrt{N}$ so that the parameter which drives the transition is the combination $\Lambda = N\alpha^2/\pi$. For example, for small times $t \ll \tau_H$, Pandey and Mehta found that the form factor is given by

$$\tilde{K}(t,\alpha) = \frac{t}{4\pi^2}[1 + e^{-4\pi\Lambda\Delta t}] \tag{3.44}$$

and interpolates between GOE ($\Lambda = 0$) and GUE ($\Lambda \sim 1$). To relate these parameters to those of our physical problem, let us return to the semi-classical description. With the above gauge transformation, the diagonal Green function gets an additional phase factor:

$$G^R(\mathbf{r}, E, \varphi) = \sum_j A_j(\mathbf{r})e^{i[S_j/\hbar + 2\pi m_j\varphi]} \tag{3.45}$$

where $\varphi = \phi/\phi_0$ and m_j is the winding number of the trajectory j. Following the same steps as in section 3.2, one finds

$$P(\omega,\varphi) = \frac{1}{2\pi\rho_0}\langle\sum_j |A_j(\mathbf{r})|^2 e^{i\omega T_j}[1 + e^{i4\pi m_j\varphi}]\rangle \tag{3.46}$$

and its Fourier transform:

$$P(t,\varphi) = \frac{1}{2\pi\rho_0}\langle\sum_j |A_j(\mathbf{r})|^2 \delta(t - T_j)[1 + \cos(4\pi m_j\varphi)]\rangle \tag{3.47}$$

The diagonal terms $k = j$ are unchanged and the terms $k = j^T$ get a phase $4\pi m_j \varphi$, since, after one turn around the loop, one trajectory picks up a phase $2\pi\varphi$ and its time reversed picks up a phase $-2\pi\varphi$. Classifying the trajectories with respect to their winding number, one finds:

$$P(t,\varphi) = P_{cl}(t,0) + P_{int}(t,\varphi)$$
$$= \frac{L}{\sqrt{4\pi D t}} \sum_{m=-\infty}^{\infty} e^{-m^2 L^2/4Dt}[1+\cos(4\pi m\varphi)] \tag{3.48}$$

The first term is the classical return probability and the second is the interference term which oscillates with period $\phi_0/2$. We have assumed that along the transverse directions, the return probability is time independent so that our expression is one-dimensional. The flux dependent part of $P(t,\varphi)$ is shown on figure 11. This figure clearly exhibits the two regimes, $t < \tau_D$ where the motion is diffusive and the flux dependence is small, and $t > \tau_D$ where the motion is ergodic and the flux dependence is large.

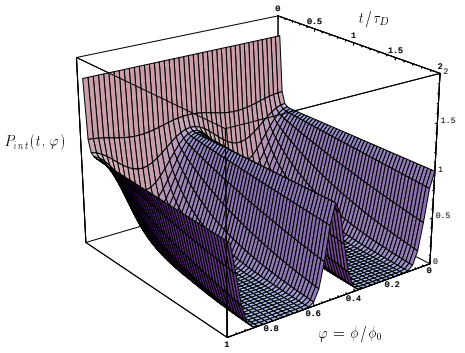

Fig. 11. Interference part of the return probability

Let us first describe the limit where $t \gg \tau_D$ and $\varphi \ll 1$ so that the sum over winding numbers can be replaced by an integral. This is the *zero-mode approximation* which corresponds to the *ergodic regime*. The return probability becomes:

$$\tilde{K}(t,\varphi) = \frac{1}{4\pi^2} t P(t,\varphi) = \frac{t}{4\pi^2}[1 + e^{-16\pi^2 E_c \varphi^2 t}] \tag{3.49}$$

This result is similar to the one found in the RMT, with the mapping:

$$\Lambda = \frac{N\alpha^2}{\pi} = 4\pi \frac{E_c}{\Delta}\varphi^2 \tag{3.50}$$

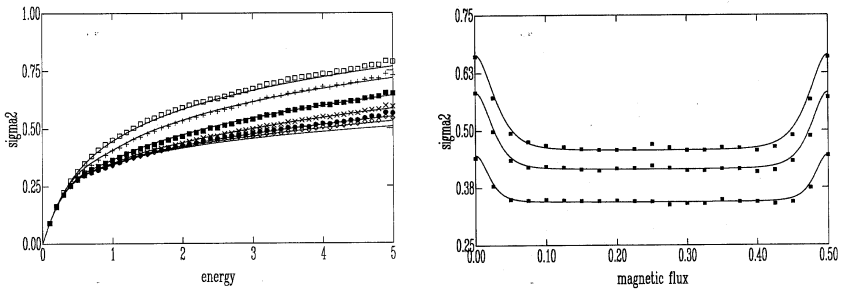

Fig. 12. Variance number $\Sigma^2(E,\varphi)$ for different values of the flux φ and comparison with RMT (*left*). $\Sigma^2(E,\varphi)$ versus flux for a given energy (*right*): note that this variation is reminiscent of the variation of $P(t,\varphi)$ vs φ when $t \gg \tau_D$, fig. 11

This mapping is quite natural: E_c/Δ, which gives the range of the RMT correlation, is related to the dimension N of the random matrices and φ plays the role of the symmetry-breaking parameter α[30].

We can then use the exact RMT result to calculate any correlation function in a flux for any energy range $\epsilon < E_c$. All the correlation functions are thus universal functions of the combination of parameters $E_c\varphi^2$[30, 58]. Using a supersymmetric approach, Altland *et al.* have indeed directly found that the two point correlation in the cross-over regime is well described by the RMT correlation function[59].

This dependence (3.49) of the form factor with the flux has a very simple origin: diffusive pairs of orbits accumulate a typical phase $\langle\Phi^2\rangle = \langle(4\pi m\varphi)^2\rangle$ where $\langle m^2\rangle$ is the typical winding number. In the diffusive regime, $\langle m^2\rangle = 2Dt/L^2 = 2E_c t$. $\langle\cos\Phi\rangle = e^{-\langle\Phi^2\rangle/2} = e^{-16\pi^2 E_c\varphi^2 t}$. The symmetry breaking depends on the length of the trajectories. For a trajectory of time t, TRS is broken for $\Phi \sim 2\pi$, i.e. $\varphi_c \sim 1/\sqrt{E_c t}$. The characteristic flux associated with an energy ϵ is thus $\phi_c \sim \sqrt{\epsilon/E_c}$.

It should be emphasized that this argument holds even for the ballistic regime in the non-integrable case, for example for billiards in an AB flux. In that case, the motion is ballistic but the accumulation of the phase with time is still diffusive[60, 61, 62, 63].

Using the relation 2.12, we obtain the number variance:

$$\Sigma^2(E) = \frac{1}{2\pi^2}\ln(1+\frac{E^2}{\gamma_\Delta^2}) + \frac{1}{2\pi^2}\ln\left(1+\frac{E^2}{(\gamma_\Delta + 16\pi^2 E_c\varphi^2)^2}\right) \qquad (3.51)$$

This fits perfectly the variation of the number variance with the flux (fig. 3.3). For $E_c\varphi^2 \gg \Delta$, the number variance is reduced by about a factor

two, as it is expected from the RMT.

In order to describe the complete variation of the form factor or of the number variance with flux, including the $\phi_0/2$ periodicity, the expression for $P(t,\varphi)$ as a discrete sum (3.48) over winding numbers m has to be used[64]. Alternatively, $P(t,\varphi)$ can also be expanded as a sum over the diffusion modes $\mathbf{q}$, (eq. 3.26). In the ring geometry, the modes are quantized by the periodic boundary conditions[64]. For a quasi-1D diffusion, $q = \frac{2n\pi}{L}$ for the classical term and $q = \frac{2\pi}{L}(n+2\varphi)$ for the interference term. As a result,

$$P(t,\varphi) = \sum_n \left(e^{-4\pi^2 n^2 E_c t} + e^{-4\pi^2 (n+2\varphi)^2 E_c t} \right) \tag{3.52}$$

There is an interesting duality between the mode (eq. 3.52) and the winding number (eq. 3.48) expansions. The zero-mode approximation corresponds to a continuous summation over the winding numbers up to infinity, as we have done above. On the other hand, the $m = 0$ winding number corresponds to an continuous summation over the modes, i.e. to an infinite system where the flux dependence is lost. These two descriptions are related by the Poisson summation formula.

For, $E > E_c$, i.e. $t < \tau_D$, a diffusive particle has a very small probability to encircle the flux. More and more diffusion modes become important and the number variance increases with the energy window as a power law. However, the flux dependence is still very small. For $E_c \gg \Delta$, it is found to vary as:

$$\Sigma^2(E,\varphi) = \Sigma^2(E,0) - \frac{1}{\pi^2} \ln(1 + 4\frac{E_c}{\gamma_\Delta} \sin^2(2\pi\varphi)) \tag{3.53}$$

One sees that in this regime, the reduction of the number variance is much smaller than what could have been naively expected. This is because the contribution to the spectral rigidity comes from short time trajectories and most of these trajectories cannot enclose a flux. This is clearly seen on fig. 11 where the flux dependence of $P(t,\varphi)$ is small.

Finally, let us also emphasize that, although the spectrum has the periodicity ϕ_0, the spectral rigidity has the periodicity $\phi_0/2$. The Hamiltonian at $\phi = \phi_0/2$ has the GOE symmetry. This is a situation of "false time-reversal violation"[65].

In the next section, we study the physics of the persistent currents which is intimately connected to the evolution of the spectral rigidity in a flux.

4. Persistent currents

4.1. Introduction

Although in the 80's most of the physics of the mesoscopic systems dealt with transport properties, a wide interest recently emerged in the experimental studies of the equilibrium properties. Among them the search for the persistent current of an isolated mesoscopic metallic ring pierced by a magnetic Aharonov-Bohm flux $\phi = \varphi\phi_0$.

Such a ring carries an equilibrium magnetization M which is the derivative of the free energy F with respect to the magnetic field H: $M = -\frac{\partial F}{\partial H}$. It corresponds to a current I flowing along the ring, which is given by

$$I(T) = -\frac{\partial F}{\partial \phi} = -\sum_n f_n \frac{\partial \epsilon_n}{\partial \phi} \qquad I(T=0) = -\frac{\partial E}{\partial \phi} = -\sum_{\epsilon_n < \epsilon_F} \frac{\partial \epsilon_n}{\partial \phi} \quad (4.54)$$

f_n is the Fermi factor associated with the energy ϵ_n. For non-interacting electrons, this current is simply the sum of the flux derivatives of each individual energy levels. Introducing the flux dependent DOS, $\rho(\epsilon, \varphi)$, the current is rewritten as:

$$I = -\frac{\partial}{\partial \phi} \int_0^{\epsilon_F} \epsilon \rho(\epsilon, \varphi) d\epsilon \quad (4.55)$$

The existence of this current has been predicted long ago by Hund[66] and first calculated by Bloch and Kulik for a clean strictly 1D ring[67, 68]. In 1983, Büttiker, Imry and Landauer proposed that it could be also observed in a metallic disordered ring[69]. This persistent current has been observed for the first time by Levy *et al.* who measured the magnetization of 10^7 isolated mesoscopic Cu rings[7]. In this experiment, the observed magnetization gives only access to the average current of each ring $\langle I \rangle$.

The estimation of this average current raises an interesting problem. From the spectrum of fig. 2 we see that the energy levels move upwards and downwards with the flux in a random way. One can thus expect single level current $i_n = \partial \epsilon_n / \partial \phi$ to be zero on average: $\langle i_n \rangle = 0$ and thus $\langle I \rangle = 0$. In other words, $\langle \rho(\epsilon, \phi) \rangle$ is flux independent. This is because the average DOS involves only trajectories of zero length so that no flux is enclosed by these trajectories. One then concludes that there is no average persistent current (the current is actually exponentially small[70, 71]. During the last years, there has been a large amount of work to explain the origin of a non-zero average current.

On the other hand, two experiments have been performed on single rings, one on Au rings[8] , the other on $GaAs - GaAlAs$ rings[9]. The relevant

quantity in these cases is the typical current $I_{typ} = (\langle I^2\rangle - \langle I\rangle^2)^{1/2} \simeq (\langle I^2\rangle)^{1/2}$.

In the following subsection, we present the calculation of the typical and average persistent current for diffusive non-interacting electrons.

4.2. The typical current

From the expression of the current in terms of the DOS, the typical current I_{typ} is given by

$$I_{typ}^2 = \frac{\partial}{\partial\phi}\frac{\partial}{\partial\phi'}\int_{-\epsilon_F}^{0}\int_{-\epsilon_F}^{0} \epsilon\epsilon' K(\epsilon-\epsilon',\varphi,\varphi')d\epsilon d\epsilon' \tag{4.56}$$

where the correlation function K is the sum of the classical and of the interference terms:

$$K(\omega,\varphi,\varphi') = K_{cl}(\omega,\frac{\varphi-\varphi'}{2}) + K_{int}(\omega,\frac{\varphi+\varphi'}{2}) \tag{4.57}$$

Similarly:

$$P(\omega,\varphi,\varphi') = P_{cl}(\omega,\frac{\varphi-\varphi'}{2}) + P_{int}(\omega,\frac{\varphi+\varphi'}{2}) \tag{4.58}$$

Fourier transforming $K(\epsilon-\epsilon')$ and using the identity $\int_0^\infty \epsilon d\epsilon e^{i\epsilon t} = -1/t^2$, one obtains straightforwardly:

$$\langle I^2\rangle = \frac{1}{2\phi_0^2}\int_0^\infty \frac{\tilde{K}''_{int}(t,\varphi)-\tilde{K}''_{cl}(t,0)}{t^4}dt \tag{4.59}$$

$$= \frac{1}{8\pi^2\phi_0^2}\int_0^\infty \frac{P''_{int}(t,\varphi)-P''_{cl}(t,0)}{t^3}dt \tag{4.60}$$

where $''$ denotes the second derivative $\partial^2/\partial\varphi^2$. The expansion of the return probability in winding numbers gives directly the harmonics decomposition of the typical current:

$$\langle I^2\rangle = \sum_{m=1}^{\infty}\langle I^2\rangle_m \sin^2(2\pi m\varphi) \tag{4.61}$$

with

$$\langle I^2\rangle_m = \frac{8m^2}{\phi_0^2}\int_0^\infty \frac{P_m(t)}{t^3}dt \tag{4.62}$$

where $P_m(t) = \frac{L}{\sqrt{4\pi E_c t}} e^{-m^2/4E_c t}$. We use ($\alpha \geqslant 1$):

$$\int_0^\infty \frac{P_m(t)}{t^\alpha} dt = \frac{2^{2\alpha-1}}{\sqrt{\pi}} \frac{E_c^{\alpha-1}}{m^{2\alpha-1}} \int_0^\infty w^{2\alpha-2} e^{-w^2} dw$$
$$= 2^{\alpha-1}(2\alpha-3)!! \frac{E_c^{\alpha-1}}{m^{2\alpha-1}}. \tag{4.63}$$

where the intermediate integral has the simple meaning of an average dimensionless winding number since $w^2 = m^2 L^2/4Dt$. Taking into account the exponential reduction $e^{-\gamma t}$ of the return probability at large times due to inelastic scattering, one finds that the harmonics are exponentially damped by a factor $e^{-m\sqrt{\gamma/E_c}} = e^{-mL/L_\varphi}$. The harmonics of the typical current are then given by:

$$\langle I^2 \rangle_m = \frac{96}{m^3} (\frac{E_c}{\phi_0})^2 [1 + \frac{m}{2} \frac{L}{L_\varphi} + \frac{m^2}{3} (\frac{L}{L_\varphi})^2] e^{-mL/L_\varphi} \tag{4.64}$$

This result should be multiplied by 4 to take the spin degeneracy into account, so that the typical current is multiplied by 2. In the limit where $L \ll L_\varphi$, the first term in brackets gives the main contribution[18, 72]. It is quite easy to understand qualitatively why the typical current is proportional to E_c (This proportionality has been first derived in ref.[70]. In that calculation, a two-cooperon diagram is missing). It is of the order of the charge e divided by the characteristic time which is nothing but τ_D: $I_{typ} \propto e/\tau_D \propto E_c/\phi_0 \propto \frac{ev_F}{L} \frac{l_e}{L}$. It turns out that the calculation of this typical current cannot explain the amplitude observed in the experiment performed on single Au rings[8]. The order of magnitude of the observed current is closer to $\frac{ev_F}{L}$. At the moment, there is no theoretical explanation for this discrepancy[73]. The experiment performed on $GaAs - GaAlAs$ rings corresponds to a situation where $l_e \simeq L$ and the observed current corresponds to the theoretical prediction.

It is also interesting to calculate the single level typical current, i_{typ}. Similarly to the typical total current, it is given by

$$i_{typ}^2 = \Delta^2 \frac{\partial}{\partial \phi} \frac{\partial}{\partial \phi'} \int_0^{\epsilon_F} \int_0^{\epsilon_F} K(\epsilon - \epsilon', \varphi, \varphi') d\epsilon d\epsilon' \tag{4.65}$$

It differs from the typical total current only by two energy factors. Therefore, this implies an additional t^2 term in the integral on time. One finds easily:

$$\langle i^2 \rangle_m = 8m^2 (\frac{\Delta}{\phi_0})^2 \int_0^\infty \frac{P_m(t)}{t} dt \tag{4.66}$$

From the general expression of the integral 4.63, we obtain

$$\langle i^2\rangle_m = 8m(\frac{\Delta}{\phi_0})^2 e^{-mL/L\varphi} \tag{4.67}$$

This expansion in harmonics diverges for $L_\varphi \to \infty$. This is because the semi-classical calculation breaks down for long times $t \geqslant \tau_H$, i.e. for large winding numbers $m \geqslant \sqrt{\Delta/E_c}$. It turns out that this expression of the typical single level current is only approximate and does not describe properly the current resulting from numerical simulations[74]. This is because, in the calculation of $\langle i^2\rangle$, all energy scales are smaller than Δ. A calculation based on the supersymmetry method is needed[75].

Finally, it is interesting to notice that a given harmonics of the typical total current is $g = E_c/\Delta$ times larger than the single level current. This means that although the total current is the sum of the currents of all the energy levels in the spectrum, the contribution of the levels nearly cancel and only the last E_c/Δ levels contribute to the total current. This is seen more precisely with a calculation of the correlation between the harmonics of the current taken at different energies: similarly to eq. 4.66, one finds

$$\begin{aligned}\langle i_E i_{E+\omega}\rangle_m &= 8m^2\left(\frac{\Delta}{\phi_0}\right)^2 \int_0^\infty \frac{P_m(t)}{t} e^{i\omega t} dt \\ &= 8m^2\left(\frac{\Delta}{\phi_0}\right)^2 \mathrm{Re}[e^{-m\sqrt{\frac{\gamma+i\omega}{E_c}}}]\end{aligned} \tag{4.68}$$

where i_E is the current at energy E. One sees that this correlation function decreases on a scale E_c/m^2[76].

4.3. The average current

We turn now to the calculation of the average current relevant for the many rings experiment[7].

4.3.1. The canonical current

The above calculation of the average persistent current, $\langle I\rangle = 0$, implicitly assumes that the Fermi level is flux independent, so that in eq. 4.55, the only flux dependence is contained in the DOS. It has been proposed that the fact that in each ring, the number of particles N is fixed and not the Fermi level, plays an important role and leads to a finite average current[77]. The reason is that the constraint that $N = \int_0^{\epsilon_F} \rho(\epsilon,\phi)d\epsilon$ is fixed implies that ϵ_F must be flux dependent. This flux dependence in eq. 4.55 being correlated to the flux dependence of $\rho(\epsilon,\phi)$ makes the average current non

zero [78]. This "canonical"* average can be found by doing an expansion around the average Fermi level ϵ_F. Denoting by $\mu(\phi)$ the sample and flux dependent chemical potential and $\epsilon_F = \langle \mu(\phi) \rangle$ one finds:

$$I_N = -\frac{\partial F}{\partial \phi}|_N = -\frac{\partial \Omega}{\partial \phi}|_{\mu(\phi)} \tag{4.69}$$

$$= -\frac{\partial \Omega}{\partial \phi}|_{\epsilon_F} - \frac{\partial^2 \Omega}{\partial \mu \partial \phi}|_{\epsilon_F}(\mu(\phi) - \epsilon_F) \tag{4.70}$$

By definition, the first term of this expression is the grand canonical current I_μ. The first derivative $-\frac{\partial \Omega}{\partial \mu}$ is the number of particles N. As a result:

$$I_N = I_{\epsilon_F} + \frac{\partial N}{\partial \phi}|_{\epsilon_F}(\mu(\phi) - \epsilon_F) \tag{4.71}$$

We then use the relation $\delta\mu|_N = -\Delta \delta N|_\mu$ which expresses that the variation of the chemical potential at fixed number of particles is proportional to the variation of the number of particles at fixed chemical potential[79]. After averaging, neglecting $\langle I_{\epsilon_F} \rangle$, one deduces the following relation[78]:

$$\langle I_N \rangle = -\frac{\Delta}{2}\frac{\partial}{\partial \phi}\langle \delta N^2(\epsilon_F, \phi) \rangle = -\frac{\Delta}{2}\frac{\partial}{\partial \phi}\int_0^{\epsilon_F}\int_0^{\epsilon_F} K(\epsilon, \epsilon', \phi) d\epsilon d\epsilon' \tag{4.72}$$

Therefore, when the number of electrons in the rings is fixed, the average persistent current is finite and is rewritten in terms of the typical sample to sample fluctuation in the number of levels below the Fermi energy ϵ_F[64, 80, 81, 82]. From the above semi-classical analysis, the current can be directly written in terms of $P(t)$:

$$\langle I_N \rangle = -\Delta \frac{\partial}{\partial \phi}\int_0^\infty \frac{\tilde{K}(t, \varphi)}{t^2} dt = -\frac{\Delta}{4\pi^2}\frac{\partial}{\partial \phi}\int_0^\infty \frac{P(t, \varphi)}{t} dt \tag{4.73}$$

The integral is known (eq. 4.63) and gives the harmonics expansion

$$\langle I_N \rangle = \sum_{m=1}^{\infty} \langle I_N \rangle_m \sin(4\pi m \varphi) \tag{4.74}$$

with

$$\langle I_N \rangle_m = \frac{2}{\pi}\frac{\Delta}{\phi_0} e^{-mL/L_\varphi} \tag{4.75}$$

* The word canonical is here slightly misleading. This calculation is actually performed in the grand-canonical ensemble, with a constraint that the average number of particles is fixed and flux independent

The average current can be reconstructed as[83]

$$\langle I_N \rangle = \frac{\Delta}{\pi\phi_0} \frac{\sin 4\pi\varphi}{\cosh L/L_\varphi - \cos 4\pi\varphi} \tag{4.76}$$

In order to take into account the spin of the electrons, the number variance should be multiplied by 4 and the interlevel spacing divided by 2 so that the current 4.76 is doubled.

This current oscillates with the period $\phi_0/2$ and is *paramagnetic* at small flux. This is clearly seen from eq. 4.72: the average current measures the change in spectral rigidity when the TRS is broken. The number variance decreases when ϕ is finite so that the current has to be paramagnetic. The order of magnitude of this average "canonical" current is smaller than observed experimentally for an ensemble of 10^7 Cu rings[7].

For completeness, we mention here that the magnetism of small disordered metallic dots – where the external parameter is a magnetic flux in the bulk of the dot instead of an AB flux – can be also described along the same ideas[84].

4.3.2. electron-electron interactions

Another larger contribution to the persistent current may come from interactions as has been first proposed by Ambegaokar and Eckern, in the framework of the Hartree-Fock(HF) approximation and using diagrammatic calculations[85, 80]. We present here a simple semi-classical derivation of this contribution to the persistent current[86]. A different approach, based on Density Functional Theory, leads to similar results[87]. The HF equations read, for each state i

$$\epsilon_i \psi_i(\mathbf{r}) = T\psi_i(\mathbf{r}) + V(\mathbf{r})\psi_i(\mathbf{r}) + \sum_j \int U(\mathbf{r}-\mathbf{r}')|\psi_j(\mathbf{r}')|^2 \psi_i(\mathbf{r}) d\mathbf{r}'$$
$$- \sum_j \delta_{\sigma_i\sigma_j} \int U(\mathbf{r}-\mathbf{r}')\psi_j^*(\mathbf{r}')\psi_j(\mathbf{r})\psi_i(\mathbf{r}')d\mathbf{r}' \tag{4.77}$$

In perturbation to the first order in the interaction parameter U, the shift of the energy levels is thus given by

$$\epsilon_i = \epsilon_i^0 + \sum_j \int U(\mathbf{r}-\mathbf{r}')|\psi_j(\mathbf{r}')|^2 |\psi_i(\mathbf{r})|^2 d\mathbf{r}d\mathbf{r}'$$
$$- \sum_j \delta_{\sigma_i\sigma_j} \int U(\mathbf{r}-\mathbf{r}')\psi_j^*(\mathbf{r}')\psi_j(\mathbf{r})\psi_i^*(\mathbf{r})\psi_i(\mathbf{r}')d\mathbf{r}d\mathbf{r}' \tag{4.78}$$

where the states ψ_i are those of the *non-interacting* system. As a result, the total energy E_T is now

$$E_T = E_T^0 + \frac{1}{2}\sum_{i,j}\int U(\mathbf{r}-\mathbf{r}')|\psi_j(\mathbf{r}')|^2|\psi_i(\mathbf{r})|^2 d\mathbf{r}d\mathbf{r}' \\ -\frac{1}{2}\sum_{i,j}\delta_{\sigma_i\sigma_j}\int U(\mathbf{r}-\mathbf{r}')\psi_j^*(\mathbf{r}')\psi_j(\mathbf{r})\psi_i^*(\mathbf{r})\psi_i(\mathbf{r}')d\mathbf{r}d\mathbf{r}' \tag{4.79}$$

where E_T^0 is the total energy in the absence of interaction. The summation $\sum_{i,j}$ is on filled energy levels. σ_i is the spin of a state ψ_i. In terms of the local density $n(\mathbf{r}) = \sum_i |\psi_i(\mathbf{r})|^2$, the total energy can be rewritten:

$$E_T = E_T^0 + \frac{1}{2}\int U(\mathbf{r}-\mathbf{r}')n(\mathbf{r}')n(\mathbf{r})d\mathbf{r}d\mathbf{r}' \\ -\frac{1}{2}\sum_{i,j}\delta_{\sigma_i\sigma_j}\int U(\mathbf{r}-\mathbf{r}')\psi_j^*(\mathbf{r}')\psi_j(\mathbf{r})\psi_i^*(\mathbf{r})\psi_i(\mathbf{r}')d\mathbf{r}d\mathbf{r}' \tag{4.80}$$

We now assume a screened Coulomb interaction: $U(\mathbf{r}-\mathbf{r}') = U\delta(\mathbf{r}-\mathbf{r}')$ where $U = 4\pi e^2/q_{TF}^2$, q_{TF} being the the Thomas-Fermi wave vector. For such a local interaction, the Fock term gets the same structure as the Hartree term and one has:

$$E_T = E_T^0 + \frac{U}{2}\int n^2(\mathbf{r})d\mathbf{r} - \frac{U}{4}\int n^2(\mathbf{r})d\mathbf{r} \tag{4.81}$$

Because of the spin, the Hartree contribution is twice the Fock contribution[85]. The interaction gives a new contribution to the persistent current

$$\langle I_{e-e}\rangle = -\langle\frac{\partial E_T}{\partial\phi}\rangle = -\frac{U}{4}\frac{\partial}{\partial\phi}\int\langle n^2(\mathbf{r})\rangle d\mathbf{r} \tag{4.82}$$

We define the local DOS $\rho(\mathbf{r},\omega)$ so that $n(\mathbf{r}) = 2\int_0^\mu \rho(\mathbf{r},\omega)d\omega$ (the factor 2 accounts for spin). The current can be rewritten as:

$$\langle I_{e-e}\rangle = -U\frac{\partial}{\partial\phi}\int\langle\rho(\mathbf{r},\omega_1)\rho(\mathbf{r},\omega_2)\rangle d\mathbf{r}d\omega_1 d\omega_2 \tag{4.83}$$

$$= -\frac{U}{2\pi^2}\frac{\partial}{\partial\phi}\int\langle G^R(\mathbf{r},\mathbf{r},\omega_1)G^A(\mathbf{r},\mathbf{r},\omega_2)\rangle d\mathbf{r}d\omega_1 d\omega_2 \tag{4.84}$$

$$= -\frac{U\rho_0}{\pi}\frac{\partial}{\partial\phi}\int P(\omega_1-\omega_2)d\omega_1 d\omega_2 \tag{4.85}$$

$$= -\frac{U\rho_0}{\pi}\frac{\partial}{\partial\phi}\int_0^\infty\frac{P(t,\phi)}{t^2}dt \tag{4.86}$$

Using again the integral (4.63), one finds easily the harmonics content of the HF current:

$$\langle I_{e-e}\rangle = \sum_{m=1}^{\infty} \langle I_{e-e}\rangle_m \sin(4\pi m\varphi) \tag{4.87}$$

with

$$\langle I_{e-e}\rangle_m = 16\frac{U\rho_0}{\phi_0}\frac{E_c}{m^2}[1+m\frac{L}{L_\varphi}]e^{-mL/L\varphi} \tag{4.88}$$

In the limit $L \ll L_\varphi$, the second term in brackets can be neglected and one recovers the result obtained for the first time with a diagrammatic calculation[85, 80, 88]. Starting from eq. 4.86, one can actually recover the expression resulting from that calculation

$$\langle I_{e-e}\rangle = \frac{U\rho_0}{\pi}\frac{\partial}{\partial\phi}\sum_q \int_0^\infty \mathrm{Re}\,\frac{-\omega}{\gamma + Dq^2 - i\omega}\, d\omega \tag{4.89}$$

The quantization of the wave vectors of the diffusion modes in now flux dependent $q(\phi)$. A Poisson summation leads to the above Fourier expansion.

Eq. 4.88 shows that the persistent current is proportional to the interaction parameter. It is known that this current is smaller than experimentally observed [7]. Moreover, inclusion of higher order contributions in the interaction parameter (the so-called Cooper channel renormalization) reduces further the amplitude of the estimated current [89, 90, 91]. It is one order of magnitude smaller than the observed current[7].

4.3.3. Local versus global fluctuations of the DOS

It is interesting to contrast the two expressions for the canonical and Hartree-Fock average currents. The canonical current results from a constraint on the conservation of the total number of particles N. From this constraint, the current has been rewritten in terms of the flux variation of the two-point correlation function of the *global* density of states. On the other hand, the HF average current has been written is terms of the two-point correlation function of the *local* DOS. This must reflect the constraint of a *local* conservation of a number of particles[80]. This can be simply understood, using a simple argument due to Argaman and Imry[87].

Suppose that the sample can be divided into pieces i in which the number of particles N_i is fixed, due to electrostatic interaction. By definition, $N = \sum_i N_i$ and the energies are extensive: $F = \sum_i F_i$ and $\Omega = \sum_i \Omega_i$. We can calculate the current in each box i as we did above for the canonical

current. The current can thus be written as

$$I_{e-e} = -\sum_i \frac{\partial F_i}{\partial \phi}|_{N_i} = -\sum_i \frac{\partial \Omega_i}{\partial \phi}|_{\mu_i(\phi)} \tag{4.90}$$

$$= -\frac{\partial \Omega}{\partial \phi}|_{\epsilon_F} - \sum_i \frac{\partial^2 \Omega_i}{\partial \mu_i \partial \phi}|_{\epsilon_F}(\mu_i(\phi) - \epsilon_F) \tag{4.91}$$

In each box, the local chemical potential μ_i has to adjust with the flux to keep N_i constant.

The number of particles in one box is $N_i = -\frac{\partial \Omega_i}{\partial \mu_i}$. Then:

$$I_{e-e} = I_{\epsilon_F} + \sum_i \frac{\partial N_i}{\partial \phi}|_{\epsilon_F}(\mu_i(\phi) - \epsilon_F) \tag{4.92}$$

Using the relation $\delta\mu_i|_{N_i} = -\Delta\delta N_i|_{\mu_i}$, averaging and neglecting $\langle I_{\epsilon_F}\rangle$, we obtain:

$$\langle I_{e-e}\rangle = -\frac{\Delta}{2}\sum_i \frac{\partial}{\partial \phi}\langle \delta N_i^2(\epsilon_F, \phi)\rangle \tag{4.93}$$

to be contrasted with the canonical current:

$$\langle I_N\rangle = -\frac{\Delta}{2}\frac{\partial}{\partial \phi}\langle \delta N^2(\epsilon_F, \phi)\rangle \quad , \quad N = \sum_i N_i \tag{4.94}$$

Replacing the discrete sum over the boxes by an integral, one finds (the interaction parameter is of order 1):

$$\langle I_{e-e}\rangle = -\frac{\Delta V}{2}\frac{\partial}{\partial \phi}\int \langle n^2(r)\rangle d\mathbf{r} \tag{4.95}$$

for the current resulting from a *local* constraint. This current has exactly the structure of the Hartree-Fock current (with a prefactor describing the strength of the interaction). On the other hand, the canonical current is

$$\langle I_N\rangle = -\frac{\Delta}{2}\frac{\partial}{\partial \phi}\int \langle n(\mathbf{r})n(\mathbf{r}')\rangle d\mathbf{r}d\mathbf{r}' \tag{4.96}$$

and results from a *global* constraint.

It is then clear from this viewpoint that the canonical current and the Hartree-Fock current have very similar physical origins, one resulting from a global conservation of the particle number while the other results from a fixed local density due to the electrostatic interaction[80].

4.4. Temperature dependence

Up to now the physical quantities have been estimated at zero temperature. The extension to finite T can be easily done by using the identity: $\int f(\epsilon)g(\epsilon)d\epsilon = 2i\pi T \sum_{\omega_n} g(i\omega_n)$ where $f(\epsilon)$ is the Fermi distribution and $omega_n = (2n+1)\pi T$. The quantities of interest (except the typical current) have the form

$$\int dE \int dE' \Gamma(E-E')f(E)f(E') = 4\pi^2 T^2 \sum_{\omega_n} \sum_{\omega'_n} \Gamma(i\omega_n - i\omega'_n) \quad (4.97)$$

$$= \int \frac{\pi^2 T^2}{(\sinh \pi T t)^2} \tilde{\Gamma}(t) dt \quad (4.98)$$

so that the t-dependent integrands involved in the different expressions of the currents should now contain the factor $(\frac{\pi T t}{\sinh \pi T t})^2$. Since the characteristic time associated with winding number m is $\tau_m = m^2/E_c$, it is clear that the temperature dependence of the m^{th} harmonics of the current is characterized by the energy scale $T_m = E_c/m^2$. This is consistent with the correlation energy which has been found in eq. 4.68, between the harmonics of different single level currents.

5. Conductance and spectrum

The conductance of a disordered system can be derived from the Kubo formula (see below). Its calculation requires the knowledge of both the wave functions and the energy levels. We would like to know whether the conductance could be also derived *only* from the knowledge of the spectrum and of its correlations. For example, the average DOS does not carry any information on the transport or on the degree of localization. But we have already seen that the conductance can in principle be obtained from the spectral analysis since $E_c = g\Delta$ is the characteristic energy scale at which the RMT correlations disappear. We have also seen that $E_c = g\Delta$ drives the flux sensitivity of the levels and that a thermodynamic quantity like the typical persistent current is proportional to the conductance g. Therefore the conductance can somehow be related to the two point-correlation function and its variation with an external parameter like an AB flux. A simple relation has been first proposed in the 70's by Thouless, which relates the conductance to the curvature of the energy levels. We analyze now this relation and more recent developments.

5.1. The Thouless formula and its extensions

A very important step linking the transport to the spectral properties has been put forward by Edwards and Thouless[12, 13] who argued that the electrical conductance can be related to the sensitivity of the energy spectrum to a *change in the boundary conditions*. Intuitively the more a level is localized, the less sensitive it is to the boundary conditions. Therefore the conductance should be directly related to a measurement of the sensitivity to the boundary conditions (we have seen that this is actually the case since the typical current is proportional to the conductance). Thouless noticed the very similar structure between the expressions of the conductance and of the curvature of energy levels when a change in the boundary conditions is introduced $\psi(x+L) = \psi(x)e^{i\eta}$, or, equivalently, an AB flux: $\eta = 2\pi\varphi = 2\pi\phi/\phi_0$. On the one hand, the d.c. $T = 0K$ conductivity σ is given by the Kubo formula:

$$\sigma = \frac{\pi e^2 \hbar}{m^2 V} \sum_{\alpha,\beta} |p_{\alpha\beta}|^2 \delta(E_F - \epsilon_\alpha)\delta(E_F - \epsilon_\beta) \tag{5.99}$$

where ϵ_α is a single energy level and $p_{\alpha\beta} = \langle\alpha|p|\beta\rangle$ is the matrix element of the momentum operator along the x- direction. On the other hand, the curvature of a given energy level α at the origin ($\eta = 0$) is easily found from perturbation theory. It is given by:

$$c_\alpha = \left(\frac{\partial^2 \epsilon_\alpha}{\partial \eta^2}\right)_{\eta=0} = \frac{\hbar^2}{mL^2} + \frac{2\hbar^2}{m^2 L^2} \sum_{\beta\neq\alpha} \frac{|p_{\alpha\beta}|^2}{\epsilon_\alpha - \epsilon_\beta} \tag{5.100}$$

In a metallic system, i.e in the presence of moderate disorder, this curvature is a random quantity. Thouless assumes first that the matrix elements $p_{\alpha\beta}$ are not correlated with the energy levels ϵ_α, so that the distribution of the curvature is roughly the distribution of the $1/(\epsilon_\alpha - \epsilon_\beta)$. Then assuming that the energy levels are not correlated, the distribution of the curvatures has the Cauchy form $P(c) = (\gamma_0/\pi)/(\gamma_0^2 + c^2)$ with a width γ_0 given by

$$\gamma_0 = \frac{2\pi\hbar^2}{m^2 L^2} \frac{|p_{\alpha\beta}|^2}{\Delta} \tag{5.101}$$

On the other hand, assuming again that $p_{\alpha\beta}$ are decorrelated from the ϵ_α, the Kubo formula gives for the average conductivity:

$$\sigma = \frac{\pi e^2 \hbar L^d}{m^2} \langle |p_{\alpha\beta}|^2 \rangle \rho_0^2 \tag{5.102}$$

$\langle ... \rangle$ represents an average over the disorder and the energy levels. Comparison between the equations 5.101 and 5.102 gives a direct relation between

the average dimensionless conductance $g = \sigma L^{d-2}/(e^2/\hbar)$ and the width of the distribution of curvatures, known as Thouless relation[13]:

$$g = \frac{1}{2}\frac{\gamma_0}{\Delta} \tag{5.103}$$

The first hypothesis is reasonable and it is correct in the framework of RMT. On the other hand, the energy levels are strongly correlated in a metal so that the second hypothesis does not hold. Therefore the curvature distribution may not have the Cauchy form, see sect. 6.1. The conductance can also be related to another quantity connected to the flux sensitivity of the levels: it has been proposed by Akkermans that the conductance could also be related to the averaged square slope of the energy levels instead of their curvature[92]. The conductance is obtained as

$$g_d = \frac{\overline{\langle i^2 \rangle}}{\Delta^2} \tag{5.104}$$

where $i(\eta) = \frac{\partial \epsilon_\alpha}{\partial \eta}$. $\overline{...}$ is an average on flux. A similar relation has also been proposed by Wilkinson[93] and by Simons *et al.*[94, 95, 96] in the case where the external parameter, instead of being an AB flux, does not break any symmetry. g_d measures a global, φ-averaged, property of the spectrum, while g given by eq. 5.103 measures a local, $\varphi \to 0$, property. It has been argued that these two quantities which are two different measures of the sensitivity of the energy levels to the boundary conditions, should actually contain the same information and should be proportional in the diffusive regime[58]. This aspect will be discussed in section 6.1.

5.2. Universal Conductance Fluctuations

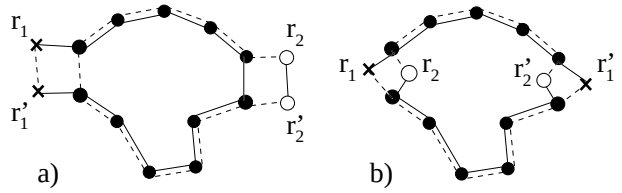

Fig. 13. Schematic view of the paths contributing to conductance fluctuations.

An important signature of the coherent nature of quantum transport is the phenomenon of Universal Conductance Fluctuations[17, 97]. When

a physical parameter is varied, such as the Fermi energy, the magnetic field or the disorder configuration, the conductance fluctuates around its average value. These fluctuations are reproducible and are the signature of the interference pattern associated to a given impurity configuration. The width of the distribution is universal and of the order of e^2/h[97]. As seen in section 1, the origin of this universality is related to the spectral rigidity of the spectrum[17]: the conductance g is equal to the number of levels $N(E_c)$ in a strip of size E_c. The spectral rigidity implies that $var[N(E_c)] \sim 1$ so that $var[g] \sim 1$. This variance can be calculated more precisely directly from the Kubo formula[17, 97]. The averaged square of the conductance contains terms of the form $\langle G^R(\mathbf{r}_1,\mathbf{r}'_1)G^A(\mathbf{r}'_1,\mathbf{r}_1)G^R(\mathbf{r}_2,\mathbf{r}'_2)G^A(\mathbf{r}'_2,\mathbf{r}_2)\rangle$ (for clarity we omitted the gradients). As shown on fig. 13, two contractions are possible: $\mathbf{r}'_1 = \mathbf{r}_1, \mathbf{r}'_2 = \mathbf{r}_2$ and $\mathbf{r}_2 = \mathbf{r}_1, \mathbf{r}'_2 = \mathbf{r}'_1$. The first term a) is proportional to $\int G^R(\mathbf{r}_1,\mathbf{r}_1,t)G^A(\mathbf{r}_2,\mathbf{r}_2,t)dtd\mathbf{r}_1d\mathbf{r}_2$. For the same reason as in section 3.2, $\mathbf{r}_1$ and $\mathbf{r}_2$ belong to the same orbit of length v_Ft. Therefore integration on $\mathbf{r}_2$ gives a factor proportional to v_Ft and the corresponding contribution to the conductance fluctuation has the form:

$$\frac{\langle \delta g^2\rangle}{\langle g\rangle^2} \propto \int_0^\infty tP(t)dt \tag{5.105}$$

This term has exactly the same structure as the two-point correlation function of the DOS. It describes the contribution of the DOS fluctuations of the conductance fluctuations[17, 98]. The second term b) is proportional to $\int P(\mathbf{r},\mathbf{r}',t)P(\mathbf{r}',\mathbf{r},\tau)dtd\tau d\mathbf{r}d\mathbf{r}'$. It can be also rewritten in the form $\int tP(t)dt$. It describes the contribution of the fluctuations of the diffusion coefficient to the conductance fluctuations[17, 98]. The integral 5.105 scales as $\tau_D^2 \propto 1/\langle g\rangle^2$. One then concludes that the fluctuations are universal.

6. Parametric correlations

6.1. Curvatures distribution

Thouless assumed that the levels are uncorrelated and found a Cauchy distribution for the curvatures. As we know from the RMT, levels are actually strongly correlated and repel each other. This repulsion must affect the curvature distribution $P(c)$. It is actually easy to find the tail of the distribution $P(c)$. When two levels are very close in energy $s \to 0$, one can isolate this pair and treat it in a perturbative way. The distance $s(\lambda)$ where λ is the perturbation parameter, varies as $\sqrt{s^2+\lambda^2} \sim s + \lambda^2/2s$ so

that for $s \to 0$, the curvature c varies as $1/s$. Since $P(s) \propto s^\beta$ one concludes that $P(c) \to 1/c^{2+\beta}$ for $c \to \infty$, in contradiction with the Cauchy form[99].

The problem of the curvature distribution has been solved recently. In the case of pure symmetry, i.e. the perturbation parameter λ does not break any symmetry (for example a step potential in the GOE case or an AB flux in the unitary case where there is already a magnetic field), the curvature distribution is given by:

$$P_\beta(c) = \frac{N_\beta}{(\gamma_\beta^2 + c^2)^{(\beta+2)/2}} \tag{6.106}$$

N_β is a normalization coefficient. This form has first been guessed by Zakrzewski and Delande[100] to fit numerical calculations on various model exhibiting chaotic spectra: the kicked-top model[101], random matrices, the kicked rotator and the stadium billiard[102], the hydrogen atom in a magnetic field. The width γ_β of the distribution is proportional to the average square of the level velocities[99, 100]:

$$\gamma_\beta = \pi\beta \frac{\langle \overline{i^2(\lambda)} \rangle}{\Delta} \tag{6.107}$$

where $i = \frac{\partial \epsilon_\alpha}{\partial \lambda}$. Afterwards, this curvature distribution has been proven analytically by Von Oppen[103] for random matrices of the form $H(\lambda) = H + \lambda K$ where H and K are random matrices belonging to the same symmetry and λ is the perturbation parameter. Recent numerical calculations have shown that this distribution is also characteristic of metallic spectra when the perturbation parameter is an AB flux ϕ[74](see fig. 14). In particular, in the limit where $\phi \to 0$, the distribution is still the GOE distribution ($\beta = 1$ in eq. 6.106)[74]. This has been proven analytically by Fyodorov and Sommers who also found that there are no corrections of order $1/g$[104].

In the literature, the coefficient γ entering the Thouless relation is sometimes taken as the typical curvature. This cannot be the case because, due to the $1/c^3$ tail, the distribution has no second moment. Instead, one can choose as a definition of the conductance[105, 106, 74, 107]:

$$g_c \equiv \frac{\langle |c| \rangle}{\Delta} \tag{6.108}$$

For pure symmetry cases, eq. 6.107 leads to universal relations between g_c and g_d[106]:

$$GOE \quad \langle |c| \rangle = \gamma_1 \quad and \quad \gamma_1 = \pi \frac{\overline{\langle i^2(\lambda) \rangle}}{\Delta} \quad \Rightarrow \quad g_c = \pi g_d \tag{6.109}$$

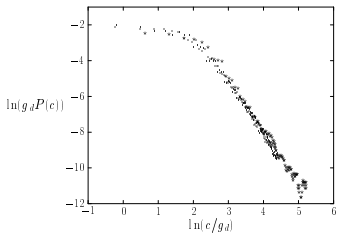

Fig. 14. After rescaling the distribution of curvatures becomes universal. ($8\times8\times8$ with $w = 4, 5, 6$)

$$GUE \quad \langle|c|\rangle = \frac{2}{\pi}\gamma_2 \quad and \quad \gamma_2 = 2\pi\frac{\overline{\langle i^2(\lambda)\rangle}}{\Delta} \quad \Rightarrow \quad g_c = 4g_d \tag{6.110}$$

The case of an AB flux has been studied recently numerically[74] and analytically with the supersymmetry method[104]. The result is that

$$g_c = 2\pi g_d \tag{6.111}$$

This relation can be simply understood: the distribution of curvatures in zero flux is characteristic of the GOE symmetry so that $\langle|c|\rangle$ is equal to the width γ of the GOE ($\beta = 1$) distribution (eq. 6.109). But there is a finite current only when $\varphi \neq 0$, i.e. when the the symmetry has become GUE, so that the relation between γ and $\langle\overline{i^2(\eta)}\rangle$ is the relation of the GUE ($\beta = 2$) symmetry (eq. 6.110). Combining these two relations, one gets $g_c = 2\pi g_d$.

When $\varphi \gg \varphi_c$, the curvature distribution becomes the one of the unitary case with a $1/c^4$ tail, and the second moment now converges. In the cross-over regime, it has been found numerically that the curvature distribution is quite different than in the pure cases, with a Gaussian tail. The typical curvature in the cross-over regime has been found to diverge logarithmically at small flux [74, 105]:

$$\langle c^2(\varphi)\rangle \propto g^2 \ln(1/g\varphi^2) \tag{6.112}$$

6.2. Parametric correlations

We have seen in section 3 that the transition between the GOE and GUE symmetries in the diffusive regime is a universal function of the combination of parameters $E_c\varphi^2$ or $g\varphi^2$.

Another quantity has been recently introduced by Altshuler *et al.* to characterize the motion of the energy levels[108, 94]. It is the auto-correlation of the current for a given level. It is defined as $C(\varphi_-) =$

$\frac{1}{\Delta^2}\overline{\langle i(\varphi)i(\varphi+\varphi_-)\rangle}$, where $\overline{...}$ is an average on flux. Since the flux drives a transition between different symmetries, the product $\langle i(\varphi)i(\varphi+\varphi_-)\rangle$ depends on both φ and φ_-. It would be translation invariant (independent of φ) in the case where the parameter φ does not break any symmetry. Here, one has to average on φ so that the correlation function is only a function of φ_-. One can use the same method as in the previous sections to calculate this correlation function. From the definition of the current and following the same lines as for the calculation of the typical current, one finds[108]

$$C(\varphi_-) = -\frac{\partial^2}{\partial\varphi_-^2}\int\int K_{cl}(\epsilon,\epsilon',\frac{\varphi_-}{2})d\epsilon d\epsilon' \tag{6.113}$$

By Fourier transform and using the relation between $\tilde{K}(t)$ and $P(t)$, one deduces[109]:

$$\begin{aligned} C(\varphi_-) &= -2\frac{\partial^2}{\partial\varphi_-^2}\int\frac{\tilde{K}(t,\varphi_-/2)}{t^2}dt \\ &= -\frac{1}{2\pi^2}\frac{\partial^2}{\partial\varphi_-^2}\int\frac{P(t,\varphi_-/2)}{t}dt \end{aligned} \tag{6.114}$$

For $\varphi_- \ll 1$, $P(t)$ has the diffusive form: $P(t,\varphi/2) = e^{-4\pi^2 E_c\varphi_-^2 t-\gamma t}$, so that

$$C(\varphi_-) = C(0)\frac{1-b\varphi_-^2}{(1+b\varphi_-^2)^2} \tag{6.115}$$

where $b = \pi^2 C(0)$ and $C(0) = \overline{\langle i^2(\varphi)\rangle} = 4E_c/\gamma$ is the average single level typical current. For large flux $\varphi_- \gg \varphi_c = 1/\sqrt{g}$, the correlation function has a universal tail[108, 110]:

$$C(\varphi_-) = -\frac{1}{\pi^2\varphi_-^2} \tag{6.116}$$

This form has also been found by Berry and Keating in the case of billiards[111]. It has the same origin, it is related to the diffusive accumulation of the phase, even when the motion is ballistic.

The expression found for $C(\varphi_-)$ does not fit correctly the numerical result for small flux. The reason is that the semi-classical calculation holds when at least one energy scale is larger than Δ, which implies here $E_c\varphi^2 > \Delta$. Thus the semi-classical calculation describes only the power-law tail of $C(\varphi_-)$, when $\varphi > \varphi_c$. On the other hand, it has been show that the small

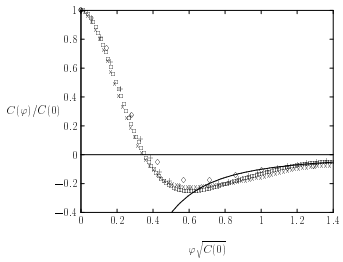

Fig. 15. Universal function $C(\varphi_-)$

flux behavior of $C(\varphi_-)$ has a logarithmic correction: $C(\varphi_-)/C(0) - 1 \propto g\varphi^2 \ln 1/g\varphi^2$[105, 74, 75, 112]. This is reminiscent of the logarithmic divergence of the typical curvature near the GOE point $\varphi = 0$.

As the GOE-GUE transition is driven by the unique combination of parameters $E_c\varphi^2$, it is seen that the parametric correlation function $C(\varphi_-)/C(0)$ is a universal function of $E_c\varphi_-^2$. This universality has been stressed by Simons *et al.* who calculated several other several parametric functions, using the supersymmetric method[94, 95, 96].

7. Conclusion

In this presentation of the spectral correlations and or their dependence on the Aharonov-Bohm flux, we have tried to unify several spectral quantities by relating all of them to the same quantity $P(t)$, the return probability for a diffusive particle. Here we summarize schematically the structure of several of these quantities, together with the weak-localization correction and with the universal conductance fluctuations. This description is based on a semi-classical picture in the diagonal approximation and fails when all energy scales become smaller than the interlevel spacing.

$K(\epsilon)$	$\rightarrow$		$\int tP(t)e^{i\epsilon t}dt$
$\Sigma^2(E)$	$\rightarrow$		$\int \frac{P(t)}{t}\sin^2(\frac{Et}{2})dt$
$\langle I^2\rangle$	$\rightarrow$	$\frac{\partial^2}{\partial\varphi^2}$	$\int \frac{P(t)}{t^3}dt$
$\langle I_{e-e}\rangle$	$\rightarrow$	$\frac{\partial}{\partial\varphi}$	$\int \frac{P(t)}{t^2}dt$
$\langle i^2\rangle$	$\rightarrow$	$\frac{\partial^2}{\partial\varphi^2}$	$\int \frac{P(t)}{t}dt$
$\langle I_N\rangle$	$\rightarrow$	$\frac{\partial}{\partial\varphi}$	$\int \frac{P(t)}{t}dt$
$\langle \Delta G\rangle$	$\rightarrow$		$\int P(t)dt$
$\langle \delta G^2\rangle$	$\rightarrow$		$\int tP(t)dt$

We have recovered in a simple way the main quantities describing the persistent currents and the parametric correlations. In these quantities, the flux dependence appears through the combination $g\varphi^2$, where g is the dimensionless conductance. This combination is the typical phase accumulated by a diffusive particle during the Heisenberg time τ_H. The observed order of magnitude of the persistent current in the experiments is still unexplained. A promising way could be a better understanding of the role of e–e interaction. Several recent works have explored this direction[114].

In this course, we have restricted ourselves to the description of level correlations in the diffusive regime. There has also been a recent activity on the study of the correlations at the Metal-Insulator transition which occurs in 3D. In the strongly insulating regime, the wave functions of two states which are close in energy do not overlap. The statistics becomes Poissonian. The Metal-Insulator transition is characterized by a third distribution, intermediate between Wigner and Poisson distributions[113].

Finally, the thermodynamic properties such as the magnetization of ballistic mesoscopic conductors, where the disorder is so weak that $l_e > L$ raises new interesting questions.

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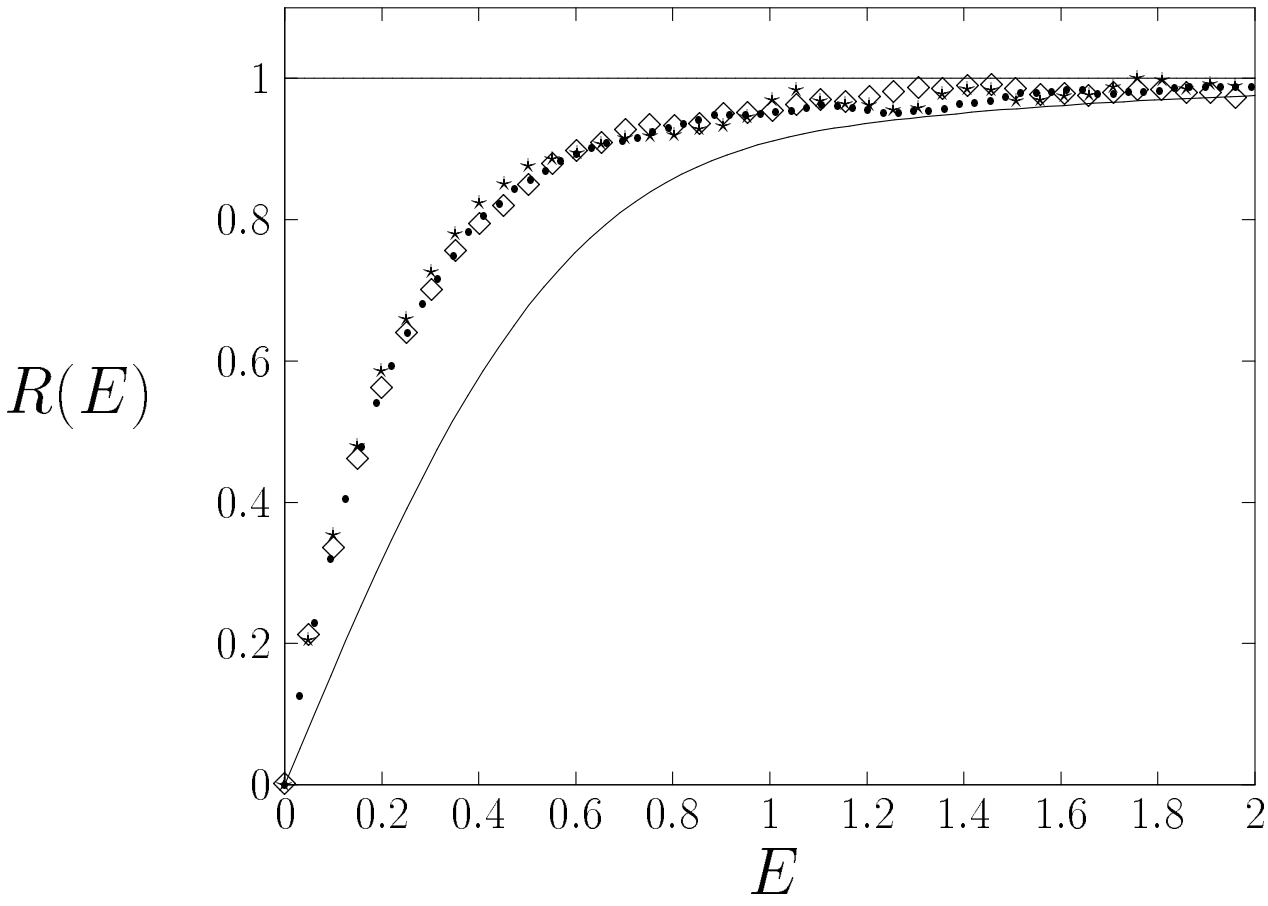
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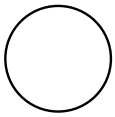

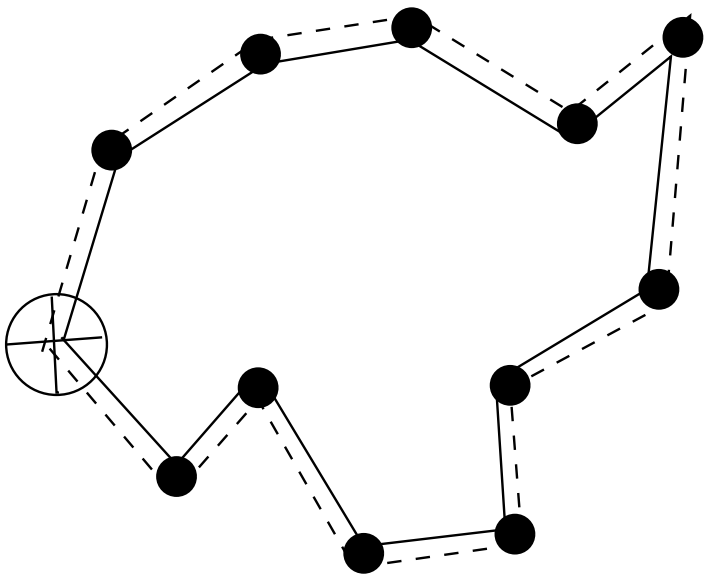

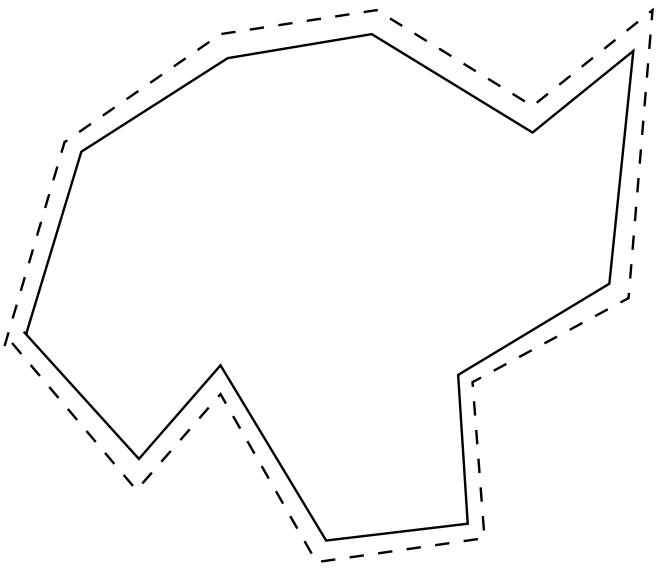

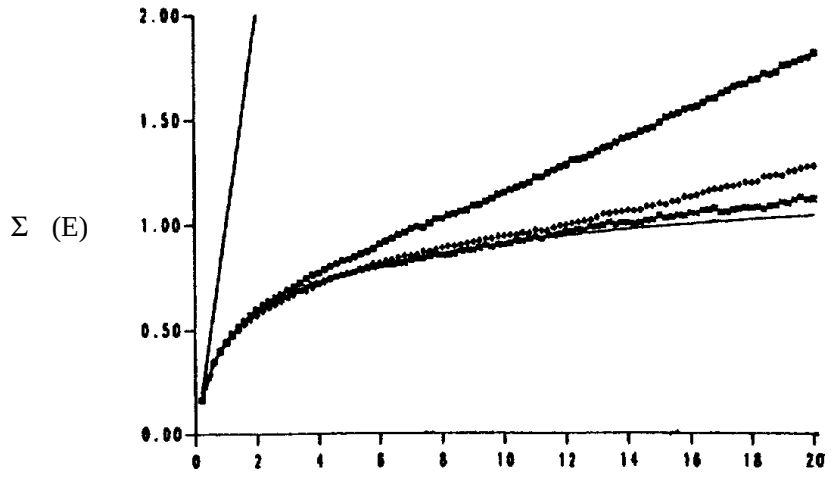
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