3.7 The Field Equations

Note. We now want a set of equations relating the metric coefficients $g_{\mu\nu}$ which determine the curvature of spacetime due to the distribution of matter in spacetime. Einstein accomplished this in his "Die Grundlage der allgemeinen Relativitätstheorie" (The Foundation of the General Theory of Relativity) in *Annalen der Physik* (Annals of Physics) in 1916.

Note. Consider a mass M at the origin of a 3-dimensional system. Let $\vec{X} = (x, y, z) = (x(t), y(t), z(t))$, and $\|\vec{X}\| = \sqrt{x^2 + y^2 + z^2} = r$. Let $\vec{u}_r$ be the unit radial vector $\vec{X}/r$. Under Newton's laws, the force $\vec{F}$ on a particle of mass m located at $\vec{X}$ is

$$\vec{F} = -\frac{Mm}{r^2}\vec{u}_r = m\frac{d^2\vec{X}}{dt^2}.$$

Therefore $\dfrac{d^2\vec{X}}{dt^2} = -\dfrac{M}{r^2}\vec{u}_r$.

Definition. For a particle at point (x, y, z) in a coordinate system with mass M at the origin, define the *potential function* $\Phi = \Phi(r)$ as

$$\Phi(r) = -\frac{M}{r}$$

where $r = \sqrt{x^2 + y^2 + z^2}$.

Theorem. The potential function satisfies Laplace's equation

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$$

at all points except the origin.

Proof. First

$$\frac{\partial r}{\partial x^i} = \frac{\partial}{\partial x^i}[(\vec{X} \cdot \vec{X})^{1/2}] = \frac{2x^i}{2(\vec{X} \cdot \vec{X})^{1/2}} = \frac{x^i}{r}$$

and

$$\frac{\partial \Phi}{\partial x^i} = \frac{\partial \Phi}{\partial r}\frac{\partial r}{\partial x^i}.$$

Therefore

$$\begin{aligned} -\nabla\Phi &= -\left(\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}, \frac{\partial \Phi}{\partial z}\right) \\ &= -\frac{M}{r^2}\left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) = -\frac{M}{r^2}\vec{u}_r = \frac{d^2\vec{X}}{dt^2}. \end{aligned}$$

Comparing components,

$$\frac{d^2 x^i}{dt^2} = -\frac{\partial \Phi}{\partial x^i}. \qquad (122)$$

Differentiating the relationship

$$\begin{aligned} \frac{\partial \Phi}{\partial x^i} &= \frac{\partial}{\partial x^i}\left[-\frac{M}{r}\right] \\ &= \frac{\partial}{\partial x^i}\left[\frac{-M}{((x^1)^2 + (x^2)^2 + (x^3)^2)^{1/2}}\right] \\ &= \frac{-1(-1/2)M(2x^i)}{((x^1)^2 + (x^2)^2 + (x^3)^2)^{3/2}} = \frac{Mx^i}{r^3} \end{aligned}$$

gives

$$\begin{aligned} \frac{\partial^2 \Phi}{(\partial x^i)^2} &= M\left(\frac{r^3 - x^i[(3/2)r(2x^i)]}{r^6}\right) \\ &= M\frac{r^3 - 3r(x^i)^2}{r^6} = \frac{M}{r^5}(r^2 - 3(x^i)^2). \end{aligned}$$

Summing over $i = 1, 2, 3$ gives

$$\nabla^2 \Phi = \frac{M}{r^5}\{(r^2 - 3(x^1)^2) + (r^2 - 3(x^2)^2) + (r^2 - 3(x^3)^2)\} = 0.$$

$\blacksquare$

Note. In the case of a finite number of point masses, the Laplace's equation still holds, only Φ is now a sum of terms (one for each particle).

Note. In general relativity, we replace equation (122) with

$$\frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0 \qquad (125)$$

where the Christoffel symbols are

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\beta} \left(\frac{\partial g_{\mu\beta}}{\partial x^\nu} + \frac{\partial g_{\nu\beta}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\beta} \right).$$

Note. Comparing equations (122) and (125), we see that

$$\frac{\partial \Phi}{\partial x^i} \quad \text{and} \quad \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$$

play similar roles. As the text says, "in a sense then, the metric coefficients play the role of gravitational potential functions in Einstein's theory."

Note. Trying to come up with a result analogous to Laplace's equation and treating the $g_{\mu\nu}$'s as a potential function, we might desire a field equation of the form $G = 0$ where G involves the second partials of the $g_{\mu\nu}$'s.

Note. "It turns out" that the only tensors that are constructible from the metric coefficients $g_{\mu\nu}$ and their first and second derivatives are those that are functions of $g_{\mu\nu}$ and the components of $R^\lambda_{\mu\nu\sigma}$ of the curvature tensor.

Note. We want the field equations to have the flat spacetime of special relativity as a special case. In this special case, the $g_{\mu\nu}$ are constants and so we desire $R^\lambda_{\mu\nu\sigma} = 0$ for each index ranging from 0 to 3 (since the partial derivatives of the $g_{\mu\nu}$ are involved). However, "it can be shown" that this system of PDEs (in the unknown $g_{\mu\nu}$'s) implies that the $g_{\mu\nu}$'s are constant (and therefore that we are under the flat spacetime of special relativity... we could use some details to verify this!).

Definition. The *Ricci tensor* is obtained from the curvature tensor by summing over one index:

$$R_{\mu\nu} = R^\lambda_{\mu\nu\lambda} = \frac{\partial \Gamma^\lambda_{\mu\lambda}}{\partial x^\nu} - \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\lambda} + \Gamma^\beta_{\mu\lambda}\Gamma^\lambda_{\nu\beta} - \Gamma^\beta_{\mu\nu}\Gamma^\lambda_{\beta\lambda}.$$

Note. Einstein chose as his field equations the system of second order PDEs $R_{\mu\nu} = 0$ for $\mu, \nu = 0, 1, 2, 3$. More explicitly:

Definition. Einstein's *field equations* for general relativity are the system of second order PDEs

$$R_{\mu\nu} = \frac{\partial \Gamma^\lambda_{\mu\lambda}}{\partial x^\nu} - \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\lambda} + \Gamma^\beta_{\mu\lambda}\Gamma^\lambda_{\nu\beta} - \Gamma^\beta_{\mu\nu}\Gamma^\lambda_{\beta\lambda} = 0$$

where

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\beta}\left(\frac{\partial g_{\mu\beta}}{\partial x^\nu} + \frac{\partial g_{\nu\beta}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\beta}\right).$$

Therefore, the field equations are a system of second order PDEs in the unknown function $g_{\mu\nu}$ (16 equations in 16 unknown functions). The $g_{\mu\nu}$ determine the metric form of spacetime and therefore all intrinsic properties of the 4-dimensional semi-Riemannian manifold that is spacetime (such as curvature)!

Note. The text argues that in a weak static gravitational field, we need

$$g_{00} = 1 + 2\Phi. \tag{135}$$

See pages 204-206 for the argument. We will need this result in the Schwarzschild solution of the next section.

Lemma III-4. For each μ,

$$g^{\lambda\beta}\frac{\partial g_{\lambda\beta}}{\partial x^\mu} = \frac{1}{g}\frac{\partial g}{\partial x^\mu} = \frac{\partial}{\partial x^\mu}[\ln|g|].$$

Proof. See pages 207-208. We will use this result in the derivation of the Schwarzschild solution.