alexander grothendieck

introduction to functorial algebraic geometry.

part 1 affine algebraic geometry.

lecture noteswritten by federico gaeta

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INTRODUCTION

TO

FUNCTORIAL ALGEBRAIC GEOMETRY

After a Summer course by

A. G R O T H E N D I E C K

VOL. I

AFFINE ALGEBRAIC GEOMETRY

Notes written by

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ALGEBRAIC GEOMETRY ONE CENTURY AGO!

Photocopy of the index of KLEIN's book:

On RIEMANN's theory of Algebraic Functions and their Integrals (1880), Dover (1963)

§ 1. *Steady Streamings in the Plane as an Interpretatio of the Functions of $x+iy$.*

The physical interpretation of those functions of x which are dealt with in the following pages is well know The principles on which it is based are here indicated, so far completeness.

Let $w=u+iv$, $z=x+iy$, $w=f(z)$. Then we have, primar

$$(1) \qquad \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x},$$

and hence

$$(2) \qquad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,$$

and also, for v,

$$(3) \qquad \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0.$$

* In particular, reference should be made to Maxwell's *Treatise on Electricity and Magnetism* (Cambridge, 1873). So far as the intuitive treatment of the subject is concerned, his point of view is exactly that adopted in the text.

Cf. DIEUDONNÉ's article D-HISTORY (cf. Bibliography) for accurate details.

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[SGA] Séminaire de géometrie algébrique: All the SGA are reprinted in Springer's collection Lecture Notes in Mathematics (LNM) with the exception of Cohomologie locale des faisceaux cohérents et Théorèmes de Lefschetz locaux et globaux (Advanced studies in pure mathematics, Masson et C^ie et North-Holland)	SGA-1-2-3-...
	SGA-2
Revêtements étales et Groupe fondamentale (LNM-224)	SGA-1
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CHAPTER 0

INTRODUCTORY MATERIAL

⌈ The impatient reader should start in CH. 0, §1, then Ch. I, §3, etc. and perhaps read the prior paragraphs only if and when he needs.

FOREWORD

These notes were primarily written from tape recordings of GROTHENDIECK's lectures during his visit at SUNYAB in the Summer of 1973. However, these recordings were supplemented by exercises, references to classical algebraic geometry, historical comments and concrete quotations of such "Bibles" as SGA, EGA, etc.[2]

GROTHENDIECK himself does not assume any responsibility for the publication of these notes; I believe however that since no adequate "textbooks" exist today and the original publications present considerable difficulties to the beginner, a publication of this kind will help a much wider audience. This is intended as an introduction to the sources SGA, EGA,...; with concrete references to Ch., § and page number, I have completed the bibliography by referring to other introductory publications such as the DIEUDONNÉ articles, MUMFORD's lecture notes, etc. Most of them contain sketchy or no proofs at all, or they are addressed to a different type of reader, cf. MACDONALD-Schemes, tailored to classical algebraic geometers). I hope that these lecture notes, directed primarily to beginning graduate students, will bridge the gap, between the previously mentioned lecture notes and the sources. To aid the newcomer, the reader will find many more details than is customary in informal publications of this type. I took advan-

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(1) *The brackets ⌈ ⌋ in text refer to my interpolations (F. Gaeta).*

(2) *The names or authors and/or titles of books, papers, etc. between " " refer to the Bibliography.*

tage of some of the oral repetitions to insert "summaries" at the beginning of most paragraphs (mostly using the tape-recorded lectures, or my own initiative if I could not find any better source). There are many complete proofs, and others are almost complete with very few, really trivial details left to the reader.

No knowledge of "old-time" or "classical" algebraic geometry was assumed although GROTHENDIECK himself gave examples involving plane algebraic curves or surfaces, etc. In many points, especially in the introduction for future applied mathematicians and in the Summary of the course, I tried to build some bridges with "old-time" algebraic geometry based on the study of algebraic varieties instead of schemes. If this might seem contrary to GROTHENDIECK's mathematical spirit, it is definitively not unfaithful to his current philosophical or sociological worries. In his prior visit to Buffalo, and in many other places as well, GROTHENDIECK campaigned against expert knowledge and technology. How can we ignore that many people feel disappointed if they do not see the words algebraic curve or surface on page one in an Algebraic Geometry text? Or they complain "a priori", just by "hearsay" that there is a lot of algebra and categorical language but - where is the geometry? I try to overcome these psychological difficulties or prejudices in order to emphasize the major simplifications introduced by GROTHENDIECK. The introduction for applied mathematicians is addressed to any person with a bachelor degree in Mathematics but it should be understood also by theoretical physicists and engineers...

I hope that very soon after a final revision of the whole course the second part dealing with the category of schemes will appear.

I am grateful to many colleagues and students in the audience who helped me in preparing these notes, mainly: J. Duskin, B. Fell, L. Gupta, R. Hansher, N. Kazarinoff, M. Klun, I. Ozaki, F. C. Lin, S. Schanuel, G. Sicherman, J. Winthrop by correcting all kinds of mistakes, typographical, linguistic, mathematical..., and I am especially grateful first of all to GROTHENDIECK who was so kind with everybody and so generous with his time. He lectured several times for periods of almost seven hours, with only a few short breaks. Who can believe that he is not interested in Mathematics anymore?

Last but not least, I am very grateful too to the typist, Mrs. Gail Berti, for her excellent job and her angelic patience, correcting and retyping the manuscript dozens of times and never once protesting.

BUFFALO, June 1974

FEDERICO GAETA

0. PROPAGANDA FOR APPLIED MATHEMATICIANS.[(1)] Not more than one century ago the distinction between pure and applied mathematics was to a large extend artificial and unimportant. For instance KLEIN's little book On Riemann's Theory of Algebraic Functions and their Integrals (1880), (Dover, 1963) (cf. sample reproduction of the index), introduced the study of Riemann surfaces by considering the practical physical problem of laminar fluid flow in a plane or arbitrary surface. He even quotes MAXWELL's treatise on page one!

The natural continuation of such "transcendental approach" in our times is the study of complex algebraic manifolds, developed by considering compact KÄHLER manifolds of the HODGE type. Although this type of analysis provides one of the most beautiful "entrances" to the edifice called Algebraic Geometry it is not considered as the most fundamental one. The main "entrance" during many years, after MAX NOETHER (around 1970) was the true "algebraic-geometric", originally related to the study of distinguished projective models of algebraic varieties and consequently to the theory of invariants. The classical paper of BRILL-NOETHER (Math. Ann., 1974) laid the foundations of "geometry on an algebraic curve" from the birational point of view.[(2)] In this approach the applicability and concreteness was still very clear and never questioned. Gradually the influence of algebra, mainly commutative algebra became more and more important and increasingly more and more abstract. The presentation of the topics became more and more detached from the applications.

[(1)] I am particularly grateful to M. Barry Fell for many valuable suggestions in writing this §0.

[(2)] Two irreducible algebraic varieties are birationally equivalent iff their fields of rational functions are isomorphic. Classical algebraic geometry considered mainly birational classes of irreducible algebraic varieties.

Today, for many colleagues, GROTHENDIECK's Algebraic Geometry looks like one of the most abstract and unapplicable products of current mathematical thought. This prejudice caused harm even before the students of mathematics within the U.S. were worried about the scarcity of academic positions... . If they ever heard GROTHENDIECK deliver one of his Survival talks aga modern Science, research, technology, etc.,... their worries might become unbearable. When h asked the audience about the usefulness of those things I recall the classical example: how c KEPLER formulate his laws on Celestial mechanics if the Greek would never study the conic sect Electromagnetism was also mentioned, and its potential harmful consequences discussed. He is very liberal man and in spite of that he allowed us to use plenty of tape recorders!

We want to show that although GROTHENDIECK's original presentation looks very abstract and seldom deals with possible applications, his inspiration is very concrete.

In contrast with Algebraic Geometry, the popular beliefs regarding Differential Geometry are totally different. Differential Geometry never lost its flavor of applicability. For propaganda, I would like to show in this introduction that such practical structures as differentiable manifolds are natural examples of <u>locally ringed spaces</u>. Thus, if a reader is acquainted with differentiable manifolds, GROTHENDIECK's schemes cannot look so terribly abstract... . It is true, we do not assume knowledge of differentiable manifolds as <u>a logical prerequisite</u> for this course, but <u>a student interested in applications should be interested in differentiable manifolds</u>.

The purpose of this informal introduction is to develop an analogy between these new mathematical objects introduced by GROTHENDIECK and certain objects within the structure of Mathematical Physics.

I will select an application which is of interest to me. Consider the "configuration sp V_n or the "phase space" W_{2n} of an holonomic dynamical system with "n-degrees of freedom"; although old books are not very precise, it is clear that for many problems concerning V_n w should only consider local functions $f: U \to \mathbb{R}$ defined within an open set $U \subseteq V_n$. For instance a Lagrangian coordinate function q_i $(i = 1,2,\dots,n)$ is only defined locally for a certain coordinate chart . The Lagrange equations of motion $\frac{d}{dt}(\frac{\partial L}{\partial q_i}) - \frac{\partial L}{\partial q_i} = 0$ are valid only in certain local coordinate systems $(q_1,\dots,q_n)$. To examine the behavior of the dynamical system globally we must piece together local functions corresponding to different o sets U.

This is achieved by first verifying that the set of functions $[f: U \to \mathbb{R} \mid U \subseteq V_n]$ form commutative ring with unit under pointwise addition and multiplication for each U. Denote ring by $\Gamma(U)$. If $V \subseteq U$ then there is a natural restriction map $r_V^U: \Gamma(U) \to \Gamma(V)$. The m r_V^U assigns to every $\varphi: U \to \mathbb{R}$ its restriction with respect to V, i.e. $r_V^U(\varphi) = \varphi|V: V \to \mathbb{R}$ In other words the local C^∞-differentiable functions on U form a "presheaf" (cf. Ch. III).

Next we must consider the "germ" of $f: U \to \mathbb{R}$ at any point $x \in U$. Let $f: U \to \mathbb{R}$ an $g: V \to \mathbb{R}$ be local functions; then f and g are equivalent functions, $f \sim g$, if they agree on s $W \subset U \cap V (\ni x)$. The germ of f at the point $x \in W$ denoted by $_xf$ is the equivalence class of functions determined by this relation. Note that this definition appears implicitly in elem "complex analysis" in one variable.

It is easy to verify that the germs $_xf$ for all $x \in W$ form a <u>local ring</u> (in the mode technical sense). Thus, with the addition of certain topological sophistications, we define "sheaf of germs of local C^∞-differentiable functions on M", denoted by $\mathcal{O}_M$ ($\mathcal{O}_X$ in the tex X a topological space), as the disjoint sum $\bigcup_{x \in M} \mathcal{O}_{M,x}$ of the local rings, $\mathcal{O}_{M,x}$, for every point of M. Thus the differentiable manifold V_n or W_{2n} of Classical Mechanics (or for that matter any differentiable manifold) is an example of a locally ringed space $(X,\mathcal{O}_x)$, i.e. a topological space X with a structure sheaf $\mathcal{O}_x$.

In spite of this heavy terminology, a sheaf of germs is really an old idea which has been made precise through a new and useful sophistication. These abstract ideas are not naturally abstract. Although they can be introduced in an abstract manner, they can be discovered "experimentally" by working with classical examples.

Sheaves were introduced to provide a transition from local to global properties. In this regard, the global study of curves which solve the classical equations of motion (a difficult problem) has been simplified by the introduction of sheaves. If we agree with LICHNÉROWICZ that the most concrete model of differentiable manifolds is the "configuration space", then sheaf theory appears to transform a concrete problem into an artificially abstract one, purely for technical or asthetic reasons. However, according to the modern approach, non-singular Algebraic Varieties can be regarded as particular cases within Algebraic Geometry when the ground field is restricted to $\mathbb{R}$ or $\mathbb{C}$. Even in the abstract case, SERRE proved in his famous paper FAC that the ZARISKI topology can be used to extend the sheaf theoretic ideas to more general abstract algebraic varieties. In this way SERRE followed H. CARTAN's idea of replacing the "field of rational functions" of an <u>irreducible</u> algebraic variety V by the sheaf of germs of local regular functions of V (which no longer need to irreducible). GROTHENDIECK's schemes are also <u>locally ringed spaces</u> $(X,\mathcal{O}_x)$.(1)

It would be dishonest to ignore here certain new complications: a differentiable manifold is HAUSDORFF, a scheme S is not even T_1 , it is just T_0 , i.e. for any couple of points $x,y \in S$ there exist an open neighborhood of one of them which do not contain the other, but this relationship is not symmetric in x,y . In other words a single point $x \in S$ is not necessarily closed: the closure $\overline{\{x\}}$ of $\{x\}$ may be very big... We shall come back to this in Vol. II. In this Vol. I we shall deal mainly with the building blocks of the schemes, the so-called <u>affine schemes</u> (Ch. III) (or affine algebraic spaces, cf. Ch. I).

1. PREREQUISITES. We shall assume familiarity with the basic algebraic structures: groups, rings, fields; the volumes of "BOURBAKI - COMMUTATIVE ALGEBRA" contain everything we are going to use. The treatise of "ZARISKI-SAMUEL", although somewhat old fashioned, is also useful. We assume familiarity with the elements of general topology, topological spaces, continuous maps, including sheaf theory. The standard reference for sheaf theory is "GODEMENT's" book. [A short introduction to this topic can be found in the short course on "TOPOI" given in SUNY at Buffalo, May 1973, inspired by SGA-4.]

In addition, the reader is supposed to be familiar with the language of category theory: (i.e. the definition of category, of functors from a category to another, the category of functors between two given categories,...and also perhaps the notion of an adjoint functor.) [The concept of a <u>representable functor</u> of a category C to the category of sets, as well as the category of covariant representable functors $C \to$ Sets, plays a big role in this course from the very beginning. I include the minimum needed to follow GROTHENDIECK's lectures in §1 of Chapter I.]

2. SUMMARY OF VOL. I

In spite of all GROTHENDIECK's revolutions, algebraic geometry is still a "geometrical theory of equations". This is made clear in Ch. I starting with a very general system of polynomial equations $S = [f_j(T_i) = 0]$ with arbitrary index sets I,J with coefficients in a ground ring k (commutative, with unit)(2). We shall consider solutions (a_i) $(i \in I)$ with coordinates a_i belonging

(1) These spaces are called geometrical spaces by DE MAZURE-GABRIEL because those which do not have an $\mathcal{O}_x$ do not seem to have enough geometrical interest...

(2) The rings considered here will be commutative rings with unit. Any ring homomorphism $f: A \to B$ preserves the unit $(f(1_A) = 1_B)$. Cf. Ch. I, §2.

any k-algebra (cf. Ch. I, §2) in particular we do not restrict ourselves
to solutions in k^I. Then to every k-algebra k' corresponds a subset
$V_S(k') \subset k'^I$ and to every homomorphism of k-algebras $k' \xrightarrow{f} k''$ corresponds
a map $V_S(f): V_S(k') \to V_S(k'')$. Thus, technically, we have a covariant
functor from the category $\mathcal{C}_k$ of k-algebras to the category of sets.[1]
Ch. I and II are devoted to the study of this functor. There are two points of
view: intrinsic and extrinsic, depending on whether or not we want to
ignore any affine embedding. Our main concern is, by far, the intrinsic
one. As a consequence two systems of equations S, S' (possibly with
$(I',J') \neq (I,J)$ are regarded as equivalent iff the corresponding functors
$V_S, V_{S'}$ are equivalent. For a fixed I, the problem is reduced to the cas
S is an ideal $\mathfrak{J}$ of the polynomial ring $P_I = k[T_i]_{i \in I}$. Then $\mathfrak{J} \neq \mathfrak{J} = V$
But if we want to avoid "the ambient affine space" $\mathbb{E}^I (= "k^I")$ we need to
duce the algebra $A_{\mathfrak{J}} = P_I/\mathfrak{J}$. Then the main result is that the intrinsic
functors $\mathcal{C}_k \to$ Sets that we are looking for are the covariant representabl
functors: $V_A: k' \mapsto \operatorname{Hom}_{k\text{-alg}}(A,k')$ attaching to a variable k' the set of
k-algebra homomorphisms from A to k'.[2] Such a functor, denoted by
V_A[3], $\mathbb{E}_A$ or simply by $\mathbb{E}$ is called the affine algebraic space over k,[4]
represented by the k-algebra A. Explicitly it is described as follows:
An "intrinsic" k'-valued point P of $\mathbb{E}_A$ is a k-alg. homomorphism

(1) This very simple example can be used to introduce the "categorical" or functorial langua
at a very early stage, since every high school graduate should know systems of equations
The reasons why V_S is a functor and not a set are discussed in Ch. I.

(2) Again, we encourage the reader with little familiarity with categories to learn the not
of representable functors in this very concrete case. The fact that we do not use any
particular property of the category $\mathcal{C}_k$ is very clear. Cf. Ch. I, §1 for further infor
mation.

(3) The V notation stands for (algebraic) "variety"; although we shall not deal systemati
with these old subjects of study of classical algebraic geometry, they still play a con
derable central role.

(4) In the case $k = \mathbb{Z}$, $\mathbb{E}$ is called an absolute (= over $\mathbb{Z}$) affine algebraic space.

$(P(a+b) = P(a) + P(b) \quad P(ab) = P(a)\cdot P(b) \quad P(1) = 1 \quad \underset{\lambda \in k}{P(\lambda a) = \lambda P(a)})$[1]

As a consequence, we have these three trivial facts:

a) <u>the set of k'-valued points</u> $\Xi_A(k')$ <u>is the same as</u> $\mathrm{Hom}_{k\text{-alg}}(A,k')$:

$$\Xi_A(k') = \mathrm{Hom}_{k\text{-alg}}((A,k')\ .$$

b) <u>the map</u> $k' \mapsto \Xi_A(k')$ <u>defines a covariant functor</u> $\mathcal{G}_A \to \mathrm{Sets}$

c) <u>to any homomorphism</u> $h\colon A \to B$ <u>of k-algebras, there corresponds a map in the opposite direction:</u>

$$h^*\colon \Xi_B(\ ) \to \Xi_A(\ )$$

<u>defined by composition in the only sensible way</u>. a), b) redefines the <u>functor of solutions</u> $k' \mapsto \Xi_A(k')$ in an intrinsic way: Ξ_A is the (covariant) representable functor $\mathcal{G}_k \to \mathrm{Sets}$ <u>represented by A</u>. c) expresses the fact that the covariant representable functors Ξ_A $(A \in \mathrm{Ob}\,\mathcal{G}_k)$ form a category equivalent to the opposite category $\mathcal{G}_k^{\circ}$ of $\mathcal{G}_k$ and the map $A \to \Xi_A$ defines a contravariant functor $\mathcal{G}_k \to \mathcal{G}_k^{\circ}$.[2]

The previous categorical properties do not tell us anything specific about the category of k-algebras[3] or its "geometrical interpretations". On the contrary, in Ch. III we shall attach <u>a geometric object</u>, the so-called <u>affine scheme</u> defined by $A = (X, \mathcal{O}_X)$ which is a particular case of a so-called <u>locally ringed space</u>,[4] to the functor Ξ_A. Before summarizing this,

[1] If this looks "too abstract" consider the embedding $\Xi_A(k') \hookrightarrow k'^{I}$, check that the "coordinates of P" are the values $P(\xi_i)$ of a system of generators and then <u>disregard the coordinates</u>... .

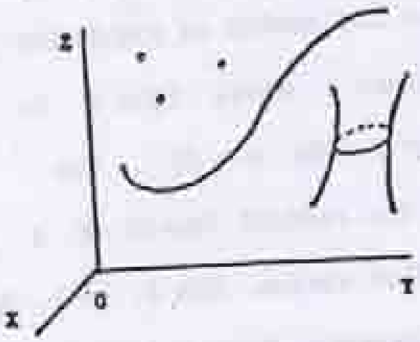
[2] This property is true <u>in any category $\mathcal{C}$</u>.

[3] See previous footnote.

[4] A ringed space $(X, \mathcal{O}_X)$ is a pair consisting of a topological space X plus a structure sheaf $\mathcal{O}_X$ on X. Most of the geometrical structures in modern mathematics are ringed spaces; for instance any kind of manifolds (topological, differentiable, analytic). They are topological spaces X with a sheaf of germs of local functions of the corresponding type (continuous, differentiable, analytic,...). "Locally ringed" means that the stalks are local rings. Cf. Ch. III, §7 for further details.

I shall very briefly recall here what the geometrical meaning of k-algebra: is, in order to firmly establish the links between old-time algebraic geometry, Geometry as a whole, and current applications.

In classical algebraic Geoemtry k is the field of complex numbers: $k = \mathbb{C}$ and we look for solutions in $\mathbb{C}^n$ $(n = \#I)$.(1) The set $V = V_S(\mathbb{C})$ i defined as an "algebraic variety" and the restrictions $f|V$ to V of polynomial functions $f: \mathbb{C}^n \to \mathbb{C}$ form a finitely generated $\mathbb{C}$-algebra A. A is isomorphic to $\mathbb{C}[T_1,\dots,T_n]/\sqrt{\mathfrak{J}}$ where $\sqrt{\mathfrak{J}}$ denotes the radical of ; (HILBERT's Nullstellensatz).(3) As a consequence A is an algebra withou non-trivial nilpotent elements. Conversely if A is a finitely generate reduced $\mathbb{C}$-algebra, A can be identified with the $\mathbb{C}$-algebra of $\mathbb{C}$-valued polynomial functions on an algebraic variety (for instance in $\mathbb{C}^3$ if V V consists of finitely many irreducible surfaces, finitely many irre



ducible curves and finitely many points. In particular V is irreducible iff A is an i domain.(4) In classical Algebraic Geometry th finitely generated $\mathbb{C}$-algebras with non trivia nilpotent elements(5) had no geometric status ever. GROTHENDIECK opposes this view, becaus

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(1) Whether J has finite or infinite elements has no importance because of the fact that the ideal $\mathfrak{J}$ generated by S has a finite basis, after HILBERT's Basissatz.

(2) The radical $\sqrt{\mathfrak{a}}$ of the ideal $\mathfrak{a}$ of A is the set: $\sqrt{\mathfrak{a}} = \{a \in A \mid \exists m \in \mathbb{Z}^+ \mid a^m \in \mathfrak{a}\}$; The radical $\sqrt{\mathfrak{a}}$ ... In particular $\sqrt{0}$ is the obviously $\sqrt{\mathfrak{a}} \supseteq \mathfrak{a}$. $\mathfrak{a}$ is a radical ideal iff $\mathfrak{a} = \sqrt{\mathfrak{a}}$. In this course: A is reduced iff $\mathrm{Nil}\,A = 0$, i.e. Nilradical of A, denoted by $\mathrm{Nil}(A)$ the only nilpotent element of A is zero. (c). 4//.

(3) The original statement (good for any algebraic closed field k) is that if f vanishes t all the "zeros" (Nullstellen) of the ideal $\mathfrak{a}$ then $f^m \in \mathfrak{a}$ for some positive integer m

(4) A commutative ring with unit is an integral domain iff $A - \{0\}$ is multiplicatively close i.e. $1 \in A - \{0\}$, $a,b \in A - \{0\} \Rightarrow ab \neq 0$.

(5) An element f of the ring A is called nilpotent iff $f^m = 0$ for some integer $m \geq 0$ (cf. footnote (2) of page 9). Applied mathematicians, engineers, etc. introduce nilpotent elements any time they disregard "infinitesimal quantities" of order $\geq h$, by writing $(x - a)^h = 0$...

algebras represent infinitesimal objects... (cf. Ch. I, §13).

Thus to help his geometrical intuition the reader should think of any k-algebra (not necessarily finitely generated over an arbitrary ground ring k, as before) as a natural generalization of the algebra of polynomial functions on V, (the problem is to "recover V from A in some sense").

If $W \subset V$, the restriction of polynomial functions defines a surjective homomorphism $A_V \to A_W$ of the corresponding rings of polynomial functions on V and W. But since non-reduced algebras had no status, classical algebraic geometry could not be "functorial" (functors were not explicitly defined, but were "used" implicitly repeating often intolerably, long statements...).

The geometrical object $(X, \mathcal{O}_X)$, the affine scheme attached to any $A \in \mathrm{Ob}\,\mathcal{C}_k$, or equivalently the affine algebraic space $\mathfrak{X}_A$ represented by A is a very powerful refinement of the old notion of algebraic variety. The space X is the spectrum of A, Spec A where Spec A is the set of all prime ideals of A) $(X = \emptyset$ iff $A = 0)$. The topology of X is defined in terms of the radical ideals of A in a manner inspired by the ZARISKI topology of affine spaces. (Cf. I, §14). Spec (A) alone does not allow us to recover A because Spec A is homeomorphic with Spec(A/Nil A). Thus, in order to construct spectra it is sufficient to restrict ourselves to reduced algebras. The structure sheaf $\mathcal{O}_X$ together with X enables us to recover A because $A \cong \Gamma(X, \mathcal{O}_X) = H^0(X, \mathcal{O}_X)$, i.e. A is isomorphic with the ring of global sections of the sheaf $\mathcal{O}_X \Leftrightarrow A$ is the 0^{th} cohomology k-algebra of X, with coefficients in $\mathcal{O}_X$). The definition of $\mathcal{O}_X$ is quite technical but we can mention here that the sta

at every point $\mathfrak{p} \in \operatorname{Spec} A$ is the local ring $A_{\mathfrak{p}}$. Actually all the

constructions make sense in the category $\mathcal{C}$ of commutative rings with uni

Note that k has no role in the construction[1] ...

CHAPTER I

FUNCTORIAL DESCRIPTION OF THE SETS OF SOLUTIONS OF SYSTEMS OF POLYNOMIAL EQUATIONS

Question:...We understand your worries about expert knowledge,...by the way, if we try to explain to a layman what algebraic geometry is it seems to me that the title of the old book of ENRIQUES is still adequate[1]: What do you think?

GROTHENDIECK's answer: Yes! but your "layman" should know what a system of algebraic equations is. This would cost years of study to PLATO..!

Question:...It should be nice to have a little faith that after two thousand years every good high school graduate can understand what an affine scheme is...What do you think?...?
.................??

From a little <u>Survival</u> talk with GROTHENDIECK.

SUMMARY. Let $S = \{f_j(T_i)\}$ $(i \in I,\ j \in J)$ indicate an arbitrary system[2] of polynomial equations $f_j(T_i) = 0$ with coefficients in a commutative ground ring with unit k . We : the set of solutions $V_S(k')$, $(k' \in \operatorname{Ob}\mathcal{C}_k)$ [3] in an arbitrary k-algebra. The map $k' \mapsto V_S$ defines a covariant functor $\mathcal{C}_k \to$ Sets. Our main problem is to characterize these functo (up to equivalence) independent of any affine embedding $V_S(k') \hookrightarrow k'^I$. The solution is functor $\mathcal{C}_k \to$ Sets is equivalent to some V'_S iff it is equivalent to some $\mathfrak{X}_A$ defined b

$$k' \mapsto \mathfrak{X}_A(k') = \operatorname{Hom}_{k\text{-alg}}(A,k') \qquad (\forall\, k' \in \operatorname{Ob}\mathcal{C}_k)$$

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(1) This looks psychologically disturbing. The ring of coefficients of our original syste equations disappears...! Not completely! There is always a homomorphism $h: \mathbb{Z} \to A$ $(n \mapsto n\cdot 1_A)$, thus A contains a ring $\mathbb{Z}/m\mathbb{Z}$ $((m) = \ker h)$ where m is the characte of A . Even the enemies of GROTHENDIECK's approach to algebraic Geometry agree tha schemes are particularly useful for arithmetic problems...<u>$\mathbb{Z}$ plays a universal rôle</u>.

(2) ENRIQUES: "<u>Geometrical theory of equations</u>...", not necessary at all to follow this :

(3) Cf. Summary of the course.

(4) Cf. §2, $\mathcal{C}_k$ denotes the category of k-algebras. For $k = \mathbb{Z}$ (ring of integers), $\mathcal{C}_{\mathbb{Z}} =$ the category of commutative rings with unit.

where $A = P_I/\mathfrak{J}$ ($\mathfrak{J} = P_I \cdot S$ is an ideal of the polynomial ring $P_I = k[T_i]_{i \in I}$). We say that $\mathbb{I}_A$ is <u>represented</u> by A, and a functor $\mathbb{G}_k \to$ Sets is equivalent to some V_S iff it is <u>representable</u>(*). (cf. §1) $\mathbb{I}_A$ is called an <u>affine algebraic space over k</u>. In particular for $A = P_I$, $\mathbb{I}_{P_I} = \mathbb{E}^I$ is the <u>standard affine space of type I</u>. It is a purely categorical fact that <u>the functors</u> $\mathbb{I}_A$ <u>are the objects of a category</u> Aff_k <u>equivalent to the opposite category</u> $\mathbb{G}_k^{*}$ of $\mathbb{G}_k$. In particular for $k = \mathbb{Z}$, $\mathrm{Aff}_{\mathbb{Z}} = \mathrm{Aff}$ is the category of "<u>absolute</u>" (i.e. over $\mathbb{Z}$) <u>affine algebraic spaces</u>, opposite to the category $\mathbb{G}$ of commutative rings with unit.

We divide Ch. I in two parts, the first part deals with the proof of $\mathrm{Aff}_k \simeq \mathbb{G}_k^{*}$. The second part deals with some particular subcategories of $\mathbb{G}$, having important geometrical and historical meaning.

REMARK. Ch. I follows the tape very closely. The contents is almost identical to the Introduction to the new Springer edition of EGA-1. No further use of this material is made in EGA-Springer.(**) A recall on representable functors and k-algebras is added in §1, 2. GROTHENDIECK started in §3.

PART I

THE ISOMORPHISM $\mathrm{Aff}_k \rightrightarrows \mathbb{G}_k^{*}$

SUMMARY. In order to characterize the V_S as <u>representable functors</u>, we establish the equivalence $V_S - V_{\mathfrak{J}}$ ($\mathfrak{J} = P_I \cdot S$) (cf. §6) and then we prove $V_{\mathfrak{J}} - \mathbb{I}_A$ ($A = P_I/\mathfrak{J}$) (cf. §7) because of the universal property of A. Conversely any $A \in \mathrm{Ob}\,\mathbb{G}_k$ can be obtained as a quotient and any $\mathbb{I}_A$ comes from some $V_{\mathfrak{J}}$.

1. REPRESENTABLE FUNCTORS. CATEGORIES OF FUNCTORS $\underline{\mathrm{Hom}}(C, \mathrm{Sets})$ AND $\underline{\mathrm{Hom}}(C^{*}, \mathrm{Sets})$. CATEGORIES S/C AND C/S.(1) Let $A \in \mathrm{Ob}\,C$ be a fixed object of the category C. The map

$$X \mapsto \mathrm{Hom}_C(A, X) \tag{1.1}$$

(*) §1 contains all the necessary prerequisites on representable functors, borrowed from GROTHENDIECK's Buffalo courses on <u>Topoi</u> and algebraic groups.

(**) The lecture notes of MANIN also start with this same approach, however MANIN presents considerably less details than in this course.

(1) This section is borrowed mainly from the Buffalo AAG-course, for several reasons mainly to avoid a "mess" confusing different hypothesis I preferred to rewrite the whole topic by myself. For further reading we recommend O-<u>Advances</u>, II, page 376 (without proofs) or <u>Fondéments</u>, page 195-01. A formal treatment is given in EGA-Springer, Ch. 0, §1, page 19. The sources are TOHÔKU and SGA, 3, I.

assigning to X the set[1] of $\mathbb{C}$-morphisms from A to X is functorial in X, i.e. it defines a covariant functor $V_A : \mathbb{C} \to \mathrm{Sets}$, which is call the <u>covariant functor $\mathbb{C} \to \mathrm{Sets}$ represented by A</u>.

In fact (1.1) defines the map $\mathrm{Ob}\,\mathbb{C} \to \mathrm{Ob}(\mathrm{Sets})$. Besides, if $X \xrightarrow{f} Y$ i a morphism in $\mathbb{C}$ then there is a natural map

$$(1.2) \qquad \mathrm{Hom}_{\mathbb{C}}(A,X) \xrightarrow{V_A(f)} \mathrm{Hom}_{\mathbb{C}}(A,Y)$$

defined by left composition with f, i.e. $u \mapsto f\circ u$, $\forall\, u \in \mathrm{Hom}_{\mathbb{C}}(A,X)$.

Similarly, we can define a contravariant functor $h_A : \mathbb{C}^{\circ} \xrightarrow{(2)} \mathrm{Sets}$ by

$$(1.3) \qquad h_A(X) = \mathrm{Hom}_{\mathbb{C}}(X,A)$$

and $h_A(f) : h_A(Y) \to h_A(X)$ for every $\mathbb{C}$-morphism $X \xrightarrow{f} Y$ where

$$(1.4) \qquad h_A(f) = v\circ f \qquad \forall\, v \in \mathrm{Hom}_{\mathbb{C}}(Y,A)$$

$h_A : \mathbb{C}^{\circ} \to \mathrm{Sets}$ is the <u>contravariant functor from $\mathbb{C}$ to Sets represente by A</u>.

REMARKS.

1) Of course either one V_A, h_A can be reduced to the other case introducing the opposite category $\mathbb{C}^{\circ}$.

2) Both types of functors V_A, h_A appear very often and naturally The main examples needed in this course arise in Algebraic Geometry whe

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(1) For any pair of objects $X, Y \in \mathrm{Ob}\,\mathbb{C}$, $\mathrm{Hom}_{\mathbb{C}}(X,Y)$ <u>is a set</u>. We did not study any foundation aspects of category theory. For some authors $\mathrm{Ob}\,\mathbb{C}$ is a <u>class</u> (not necessarily a set); or following GROTHENDIECK (SGA 3), it can be a big set satisfying certain properties which is called a <u>universe</u>.

(2) $\mathbb{C}^{\circ}$ is the opposite category of $\mathbb{C}$ i.e. it has the same objects and the same arrows as $\mathbb{C}$ with the direction and the order of composition of arrows reversed.

$\mathbb{C}$ is the category $\mathfrak{G}_k$ of k-algebras (where k is a fixed ground ring, commutative with unit) (cf. §2) or its dual $\mathfrak{G}_k^{*}$[1]. In fact <u>the elements</u> (ring homomorphisms) $u \in \mathrm{Hom}_{\mathfrak{G}_k}(A,k')$ $(k' \in \mathrm{Ob}\,\mathfrak{G}_k)$ <u>are k'-valued points</u>[2] <u>of the affine algebraic space represented by A</u>.

A covariant (contravariant) functor F from $\mathbb{C}$ to Sets is called <u>representable</u> by the object $A \in \mathrm{Ob}\,\mathbb{C}$ iff F is equivalent to some V_A (h_A. If this is the case <u>the representing object A</u> is determined up to isomorphism. In order to make clear this statement we need to formalize the previous definition by introducing the categories <u>Hom</u>($\mathbb{C}$,Sets) (<u>Hom</u>(($\mathbb{C}^{*}$, Se of covariant (contravariant) functors from $\mathbb{C}$ to Sets. According to well-known recipes[3] an object of <u>Hom</u>($\mathbb{C}$,Sets) is a covariant functor: $F\colon \mathbb{C} \to$ Sets. If $F,G \in \mathrm{Ob}$ (<u>Hom</u> ($\mathbb{C}$,Sets)) a morphism $u\colon F \to G$ is a natural transformation, i.e. for every $A \in \mathrm{Ob}\,\mathbb{C}$ there exists a map

$$u(A)\colon F(A) \to G(A)$$

such that for every $\mathbb{C}$-morphism $f\colon A \to B$ we have a commutative diagram:

(1.5)
$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "u(A)"'] & F(B) \arrow[d, "u(B)"] \\ G(A) \arrow[r, "G(f)"] & G(B) \end{tikzcd}$$

In particular u is an equivalence if the vertical arrows $u(A)$ are <u>equivalences for any choice of $A \in \mathrm{Ob}\,\mathbb{C}$</u>.

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(1) <u>Precisely the fact that $\mathfrak{G}_k^{*}$ can be identified naturally with the category of the so-called affine algebraic spaces over k is the main result of this Ch. I.</u>

(2) Naively: "points with coordinates in k' (we do not want coordinates...).

(3) If $\mathbb{C}$,$\mathbb{S}$ are two categories Funct($\mathbb{C}$,$\mathbb{S}$) = <u>Hom</u>($\mathbb{C}$,$\mathbb{S}$) (<u>the category</u> of covariant functors $\mathbb{C} \to \mathbb{S}$) has as morphisms the natural transformations.

We leave to the reader the case of $\underline{\mathrm{Hom}}\ (C^{\circ}, \mathrm{Sets})$, i.e. of the categc
of Set-valued contravariant functors).

The category $\underline{\mathrm{Hom}}\ (C^{\circ}, \mathrm{Sets})$ will be denoted by $\hat{C}$. Consequently
$\hat{C}^{\circ} = \underline{\mathrm{Hom}}\ (C, \mathrm{Sets})$. $\hat{C}$ is a natural enlargement of C obtained by identi
fying any object $X \in \mathrm{Ob}\, C$ with the contravariant functor h_X "represente
by X"; h_X is the contravariant functor from C to sets defined by
$h_X(Y) = \mathrm{Hom}_C(Y,X)$. In fact, the embedding functor

$$i : C \hookrightarrow \hat{C} \tag{1.6}$$

defined by

$$i(X) = h_X \quad \forall\, X \in \mathrm{Ob}\, C \tag{1.7}$$

is a fully faithful functor[1] from C to $\hat{C}$ which enables us to consi
C as a full subcategory[2] of $\hat{C}$. The objects of the essential image(
of C by i are the so-called contravariant representable functors.

The covariant representable functors, used extensively in this Ch.
are obtained by applying the $\hat{}$-construction to the opposite category C°
In other words: we can define an embedding

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(1) A functor $F : C \to \mathcal{B}$ is faithful iff for every pair of objects $A, B \in \mathrm{Ob}\, C$ the induced
$\mathrm{Hom}(A,B) \to \mathrm{Hom}(F(A),F(B))$ is injective. A faithful functor is called fully faithful i
for every choice of A, B the previous map is bijective. If F is faithful, $F(C)$ is
subcategory of $\mathcal{B}$. If F is fully faithful the category, $F(C)$ is a full subcategor
$\mathcal{B}$. (cf. footnote (2)).

(2) Let X, Y be two objects of a subcategory $\mathcal{B}$ of a category $\mathcal{G}$. Then we have a natur
injection $\mathrm{Hom}_{\mathcal{B}}(X,Y) \hookrightarrow \mathrm{Hom}\ (X,Y)$, i.e. every morphism in $\mathcal{B}$ is also a morphism in $\mathcal{G}$
$\mathcal{B}$ is a full subcategory of $\mathcal{G}$ iff for every pair of objects $X, Y \in \mathrm{Ob}\, \mathcal{B}$ the previou
inclusion is an equality:

$$\mathrm{Hom}_{\mathcal{B}}(X,Y) = \mathrm{Hom}_{\mathcal{G}}(X,Y)$$

(3) Let $i : \mathcal{C} \to \mathcal{J}$ be a fully faithful functor from $\mathcal{C}$ to $\mathcal{J}$. The essential image of
$\mathcal{J}$ is the full subcategory (cf. footnote (3)) of $\mathcal{J}$ whose objects are equivalent to
the image $i(\mathcal{C})$, i.e. $F \in \mathrm{Ob}\, \mathcal{J}$ belongs to the essential image of $\mathcal{C}$ iff $\exists X \in \mathrm{Ob}$
that $F \cong i(X)$.

$$j : C^{\circ} \to \hat{C}^{\circ} \tag{1.8}$$

of C in the category of covariant functors from C to sets by $j(X) = j_X$ being defined as $Y \mapsto j_X(Y) = \mathrm{Hom}_C(X,Y)$. <u>The covariant representable functors from C to Sets are those of the essential image of C° by j</u> .

We can summarize these considerations as follows:

There are functors $h_X = \mathrm{Hom}_C(\ ,X)$ and $j_X = \mathrm{Hom}_C(X,\ )$ from C to Sets of type

$$Y \mapsto h_X(Y) = \mathrm{Hom}_C(Y,X) \qquad Y \mapsto j_X(Y) = \mathrm{Hom}_C(X,Y) \tag{1.9}$$

and transforming C-morphisms $Y \xrightarrow{f} Z$ by composition in the only possible way. Thus h_X is a <u>contravariant functor</u>: $C^{\circ} \to$ Sets and j_X is a <u>covariant functor</u> $C \to$ Sets. We say in both cases that h_X (resp j_X) is a contravariant (covariant) functor from C to Sets <u>represented by X</u> . More generally:

A contravariant (covariant) functor F from C to Sets is called <u>representable</u> iff there exists an object $X \in \mathrm{Ob}\,C$ such that $F \simeq h_X$ (or $\simeq j_X$) . In both cases we say that <u>X is an object of C representing the functor F</u> . It is clear that if $X' \simeq X$ in C then X' represents F iff X represents F . In other words: <u>the representing object X is defined up to isomorphism.</u>(1)

REMARK. In most cases a contravariant (covariant) functor F represented by X will be identified with h_X (or j_X) on the precedent notions will be sufficient.(2)

(1) We leave the easy verification to the reader.

(2) This doesn't cause any problem as long as we are not concerned about unique isomorphism.

In the more sophisticated questions it is necessary to emphasize the choice of a distinguished element $\xi \in F(X)$ which is the image by the set equivalence γ_X: Sets $\to$ Sets of the identity $1_X \in \mathrm{Hom}_C(X,X) = h_X(X)$ $(= j_X(X))$. In fact in the contravariant case a morphism $Y \xrightarrow{f} Z$ in C induces a commutative square

$$\text{(1.10)} \qquad \begin{array}{ccc} h_X(Z) & \xrightarrow{h_X(f)} & h_X(Y) \\ \gamma_X(Z)\downarrow & & \downarrow\gamma_X(Y) \\ F(Z) & \xrightarrow{F(f)} & F(Y) \end{array} \qquad\qquad \begin{array}{c} 1_X \\ \gamma_X(X)\downarrow \\ \xi \end{array}$$

and, in particular, for $Z = X$ we have a map $h_X(f): h_X(X) \to h_X(Y)$ such that $(h_X(f))(1_X) = f$. ξ has the following universal property:

$$\text{(1.11)} \qquad (\gamma_X(Y))(f) = (F(f))(\xi)$$

As a consequence when we distinguish F from its equivalent h_X by means of the concrete equivalence: $Y \mapsto \gamma_X(Y)$

$$\text{(1.11)} \qquad \gamma_X(Y): h_X(Y) \to F(Y)$$

we can say that F is completely determined by the representing object $X \in \mathrm{Ob}\, C$ and the distinguished element $\xi = (\gamma_X(X))(1_X)$.

We leave to the reader the consideration of ξ in the covariant c as well as the corresponding conclusion that F is uniquely determined X and the universal element $\xi \in F(X)$.

From this more concrete point of view becomes more correct to defi

F as representable iff there exists a couple (X,ξ) $(X \in \mathrm{Ob}\,\mathbb{C}$ and $\xi \in F(X))$ such that ξ has the universal property quoted above.

In the previous paragraphs the object X was fixed. Now we are going to let X move in the class $\mathrm{Ob}\,\mathbb{C}$ and we shall consider the previous representable functors as objects of two new categories (whose morphisms are the natural transformations) denoted by GROTHENDIECK with the generic name of <u>base changes</u> (for both variancies). Let us distinguish the two cases:

<u>Contravariant case</u>: $Y \to \mathrm{Hom}(Y,X) = h_X(Y) \quad \forall Y \in \mathrm{Ob}\,\mathbb{C}$

These contravariant representable functors for a variable $X \in \mathrm{Ob}\,\mathbb{C}$ are the objects of a category naturally isomorphic to $\mathbb{C}$ (with canonical isomorphism $X \mapsto h_X$. This is a <u>fully faithful functor</u> which allows us to identify $\mathbb{C}$ with a full subcategory of the category $\hat{\mathbb{C}} = \underline{\mathrm{Hom}}(\mathbb{C}^{\circ}, \mathrm{Sets})$ of contravariant functors from $\mathbb{C}$ to Sets.(1)

<u>Covariant case</u>: $X \to \mathrm{Hom}(S,X)$

Similarly the covariant representable functors j_X can be identified with X regarded as objects of the opposite category $\mathbb{C}^{\circ}$.(2) In other words: <u>The opposite category $\mathbb{C}^{\circ}$ of $\mathbb{C}$ is canonically isomorphic with the category of covariant representable functors.</u>

Let X be a fixed object of a category $\mathbb{C}$; we are going to define two categories $S/\mathbb{C}$, $\mathbb{C}/S$ whose objects are $\mathbb{C}$-morphisms of source (target) S respectively and whose morphisms are $\mathbb{C}$-arrows making the corresponding triangles commutative

(1) This canonical embedding $\mathbb{C} \rightarrowtail \hat{\mathbb{C}}$ enabled GROTHENDIECK in TOHÔKU (Cf. Ch. II, §1,2) to reduce the $\varprojlim$ constructions to the set theoretic case.

(2) j can be used to define the $\varinjlim$ constructions (Cf. Ch. II, §3,8).

$u \in Ob(S/C) \Leftrightarrow u: S \to X_u$

$$\varphi \in Arr(S/C) \Leftrightarrow \begin{tikzcd} S \arrow[r] \arrow[dr, "u"'] & X_u \arrow[d, "\varphi"] \\ & X_v \end{tikzcd}$$

$u = \varphi \cdot v$

$u \in Ob(C/S) \Leftrightarrow u: X_u \to S$

$$\psi \in Arr(C/S) \Leftrightarrow \begin{tikzcd} X_u \arrow[r] \arrow[d, "\psi"'] & S \\ X_v \arrow[ur, "v"'] & \end{tikzcd}$$

$u = v \cdot \psi$

where as usual $Ob\,C$, $Arr\,C$ denote the classes of <u>objects</u> and of <u>arrows</u> of C. An object ϕ_C of C is called an <u>initial object of C</u> iff for every $X \in Ob\,C$ there exists just one morphism $\phi_C \to X$. Obviously any two initial objects of C are isomorphic. The category of Sets has as initial object the empty set.

Dually an object e of C is called a final object iff for every $X \in Ob\,C$ there exists just one morphism $X \to e$. One-point sets are final objects in the category of sets. Any two final objects are isomorphi

Of course an initial (final) object of C becomes final (initial) in

If C has an initial (final) object $\phi_C(e)$ we can identify C with ϕ_C/C (or C/e) in an obvious way.

2. THE CATEGORY $\mathbb{G}_k$ OF k-ALGEBRAS.

Our starting point will be the study of soluti of a very general system of polynomial equations with coefficients in a commutative ground ring with unit k. But we cannot restrict ourselves to solutions in k, but rather we shall look for solutions in an arbitrary k-algebra k'. We shall recall what these k-algebras mean from a categorical point of view.

Let $\mathbb{G}$ be the category of commutative rings with unit; a morphism (or "arrow") in $\mathbb{G}$ is a unit-preserving ring homomorphism $f: A \to B$ $(A, B \in Ob\,\mathbb{G})$, i.e. f satisfies the following conditions (2.1), (2.2), (

(2.1), (2.2) $f(x+y) = f(x) + f(y) \qquad f(xy) = f(x)f(y)$

(2.3) $$f(1_A) = 1_B$$

The ring $\mathbb{Z}$ of integers is an initial object of $\mathcal{C}$. (Cf. §1) In fact, any arrow $\varphi\colon \mathbb{Z} \to A$ maps $\pm(1 + 1 + \overset{m}{\cdots} + 1)$ in $\pm(m\cdot 1_A)$, thus φ is unique and determined by the condition (2.3). φ is not necessarily injective. $\varphi(\mathbb{Z}) \simeq \mathbb{Z}/m\mathbb{Z}$ where $m(\geq 0)$ is the characteristic of A. If A is a field, then $m = 0$ or any prime number.

Let k be any ring(*) of $\mathcal{C}$. The objects of the category of morphisms $k \xrightarrow{h} A$ (in $\mathcal{C}$) are usually called, for short k-algebras, i.e. according to general categorical procedures we construct the category $\mathcal{C}_k$ of k-algebras as follows:

1) An object of $\mathcal{C}_k$ is a morphism $k \xrightarrow{h} A$.

Usually by abuse of language A is called a k-algebra and k is called the structural morphism. We should keep in mind that h should be known.

2) A morphism of $k \xrightarrow{u} A$ in $k \xrightarrow{u} B$ in the category $\mathcal{C}_k$ is a morphism $\varphi\colon A \to B$ such that the following triangle

(2.4)
$$\begin{tikzcd} & k \arrow[dl] \arrow[dr] & \\ A \arrow[rr] & & B \end{tikzcd}$$

commutes.

Since, as we said, the morphism u,v are "clear", the following terminology is widely used:

To $k \xrightarrow{h} A$ corresponds an external product $k \times A \to A$ defined by

(*) "Ring" means always an object of $\mathcal{C}$, i.e. a commutative ring with identity.

(2.5) $$(\lambda, a) \to (h(\lambda))a = a(h(\lambda))$$

In particular since $h(1_k) = 1_A$ $(1,a) \to a$ and this scalar multiplication $k \times A \to A$ is added to $(+,\cdot,1_A)$ as an extra-structure in such a way that the following properties

(2.6) $$\lambda(a+b) = \lambda a + \lambda b$$

(2.7) $$\lambda(ab) = (\lambda a)b = a(\lambda b)$$

are satisfied. Using this approach the category $\mathbb{G}_k$ of k-algebras contai as objects the k-algebras, i.e. rings $k' \in \mathrm{Ob}\,\mathbb{G}$ endowed with a structural morphism $h: k \to A$ and a morphism $\varphi: A \to B$ of k-algebras is a morphism of $\mathbb{G}$ commuting with scalar multiplications :

(2.8) $$\varphi(\lambda a) = \lambda\varphi(a) \qquad a \in A \quad \lambda \in k$$

Since, for a given k the k-algebras are particular cases of rings it may look at first sight that the study of k-algebras would be "more restrictive". It is not so however, because of the fact that $\mathbb{Z}$ is an initial object of $\mathbb{G}$ and every unit preserving ring homomorphism $A \to B$ commutes with $\mathbb{Z}$-multiplications. In other words,

The category $\mathbb{G}$ of commutative rings with unit and unit preserving ring homomorphisms is identical with the category $\mathbb{G}_{\mathbb{Z}}$ of $\mathbb{Z}$-algebras and $\mathbb{Z}$-homomorphisms.

3. IDENTIFICATION OF POINTS IN k'^I WITH HOMOMORPHISMS OF k-ALGEBRAS.

SUMMARY. If $k' \in \mathrm{Ob}\,\mathbb{G}_k$ (cf. §1), and I is an arbitrary index set, any point $x \in k'^I$ (the standard affine space of type I over k') defines a homomorphism ϵ_x of k-algebras: $\epsilon_x: P_I \to k'$ of the polynomial ring $P = P_I = [f(T_i)_{i \in I}]$ where T_i is a family indexed by I

of algebraically independent indeterminates adjointed to k. Conversely, <u>any such homomor-</u>
<u>phism u determines a unique point</u> $(u(T_i))_{i \in I} \in k'^I$.

Let us begin by considering the usual state of affairs in algebraic geometry or in arithmetic: let k be a commutative ring with unit. In the classical situation k is the field $\mathbb{R}$ or $\mathbb{C}$ of real or complex numbe but in many arithmetic problems k may be the ring $\mathbb{Z}$ of integers or the ring of integers in a number field... . In any event, let us fix k for the time being and will be referred to as <u>the ground ring</u>.

Let $P = P_I$ be the ring of polynomials with coefficients in k in a set of indeterminates T_i indexed by a set I (*),

(3.1)
$$P = P_I = k\,[(T_i)_{i \in I}]$$

Obviously $P_I \in \mathrm{Ob}\,\mathfrak{A}_k$, i.e. P_I is a k-algebra generated by the T_i, $i \in I$.

Let $x = (x_i)_{i \in I}$ $(x_i \in k, \forall i \in I)$ be a family of elements of k indexed by I. x is called <u>a point of the standard affine space k^I (1) of type I</u>.

Let $f \in P_I$ be a polynomial; f defines a function $k^I \to k$, which sometimes is denoted with the same letter, although it might not determine back f uniquely. Let us call it now $\tilde{f}$ to avoid abuse of language: $\tilde{f}\colon k^I \to k$ is defined by assigning to every $x \in k^I$ the element $f(x)$ of k obtained by replacing the indeterminate T_i by x_i for every $i \in I$. More generally we can define $\tilde{f}_{k'}$, also as a function $k'^I \to k'$ where k'

(*) We do not assume I to be necessarily finite, although in many classical cases $I = \{i\}$, $\{1,2\}$, $= \{1,2,3\}, \ldots$ (set of equations in the affine line, plane, 3-space...). This generality is justified later by technical reasons.

(1) k^I is the power set = set of all the maps: $I \to k$. Of course if $I = \{1,2,\ldots,n\}$ the notation is shortened by k^n. In GROTHENDIECK words: $\mathbb{E}^I$ is the Cartesian power of exponent I of the forgetful functor $\mathbb{E}^1$ which associates to every $k' \in \mathrm{Ob}\,\mathfrak{A}_k$ its underlying set.

is any k-algebra.

Let us fix $x \in k'^I$! Then the map $i \mapsto x_i$ $(\in k')(\forall\ i \in I)$ define
a homomorphism $P_I \xrightarrow{u} k'$ of k-algebras by $f(T_i) \mapsto f(x_i)$ cf. §2, 12.8
that can also be written as $x \mapsto \varepsilon_x$ $(x \in k')^I$, $\varepsilon_x \in \mathrm{Hom}_{k\text{-alg}}(P_I,k')$ wh
the image $\varepsilon_x(f)$ of the polynomial $f \in P_I$ by the (k-alg)-homomorphis
ε_x is the element $f(x_i)$ obtained from f by replacing T_i by x_i ,
$\forall\ i \in I$. In fact, we can immediately verify the characteristic proper
of a (k-alg.)-homomorphism for every fixed point x of the affine spa
Conversely any $u \in \mathrm{Hom}_{k\text{-alg}}(P_I,k')$ defines a point of k'^I, i.e. it

4. SOLUTION SETS $V_S(k')$ WITH COORDINATES IN A k-ALGEBRA.

⌈SUMMARY. We disregard the arithmetic problem of looking for k-valued solutions of ou system S of polynomial equations, as being "too difficult". Some justifications are gi for the easier algebraic geometric problem, studied by KAHLER and GROTHENDIECK of looking solutions in arbitrary k-algebras.⌋

Consider a certain family of polynomials $(f_j)_{j \in J}$, where J is another fixed arbitrary set of indexes. The usual interpretation of the "system S of polynomial equations"

$$(4.1) \qquad S) \qquad f_j(x) = 0 \qquad j \in J$$

comes to mind. Our first thought about the problem of investigating solutions of such equations consists in looking for the points $x \in$ I such that $\bar{f}_j(x) = 0$ $\forall\ j \in J$; this point of view arose in the clas case of the affine plane over $\mathbb{R}$, $\mathbb{C}$.(1) When k is fixed in this

(1) For instance, let us assume that S consists of the equations of a line L and a ci C in the Euclidean real plane. If L and C do not meet we "accept" the two compl solutions $\in \mathbb{C}^2$,... . Intersection problems, lead from $\mathbb{R}$ to $\mathbb{C}$ at the very begin of classical algebraic geometry.

number of problems arise, which may be deep and hard to solve, for instance:

1) Does any solution of the system S, exist?

2) If there is any solution, are the total number of solutions finite or infinite?

3) If the former case, can we give some estimate, either exact or approximate, of the number of solutions?

4) In the latter case, the number of solutions might become finite if we add certain inequalities, or can we give asymptotic formulas for these estimates when the parameters assume certain limit values. For instance, in the plane let us look for solutions contained in a certain square centered at the origin with sides parallel to the axes and let us ask about the asymptotic estimates when the length of side converges to ∞...

The answers that we obtain to these questions will be extremely different according to the nature of the ring k. For instance comparing $\mathbb{R}$, $\mathbb{C}$ the fact that $\mathbb{C}$ is algebraically closed and $\mathbb{R}$ is not makes a big difference. Further, if k is a finite field or if we take k to be the ring of integers the kind of answers we obtain are considerably different. These problems are the hardest. This is arithmetic. We are going to look at a somewhat different approach by allowing the solutions to vary not only in k^I, i.e. "the standard affine space of type I"; but in any k'^I where k' is any k-algebra; and we shall look at the sets $V_S(k')$ of solutions of this system of algebraic equations for variable k' as a functor $k' \mapsto V_S(k')$ with respect to the variable k'. Affine algebraic geometry, roughly speaking, will be the study of such (and closely related functors).

⌈ The classical device of passing from real solutions to complex solutions for systems of algebraic equations with real coefficients can be considered as the first step in this direction ($k' = \mathbb{R}$, $\mathbb{C} = \mathbb{R}[1,i]$, ..., Cf. footnote (1) in page 23). However, classical geometers always thought of keeping fixed some field: first $\mathbb{C}$, then any algebraically closed field, accepting variable "definition fields" in a remedial basis. Without the functorial language, the description of the solutions with coefficients in arbitrary k-algebras, due to KÄHLER, and then especially to GROTHENDIECK, would be very cumbersome.⌋

5. THE FUNCTOR $V_S: \mathbb{G}_k \to$ Sets DESCRIBING THE SOLUTIONS OF S.

SUMMARY: Although affine embeddings are avoided, very often, we define first the functor $\mathbb{E}^I$ (affine linear space of type I), from k-algebras to sets associating to any $k' \in \mathrm{Ob}\,\mathbb{G}_k$ the affine space $\mathbb{E}^I(k') = k'^I$ with coordinates in k': $k' \to k'^I$.

Let S be an arbitrary system of polynomial equations of type (I,J) with coefficients in the ground ring k. The functor V_S of solutions ($\Leftrightarrow$ "algebraic variety defined by k) can be introduced first as a subfunctor of $\mathbb{E}^I$: $V_S \hookrightarrow \mathbb{E}^I$. V_S maps every k-algebra k' into the set of solutions of S in the affine space $\mathbb{E}^I(k')$. (Later we shall get rid of $\mathbb{E}^I$).

Using GROTHENDIECK's own words:

...For a variable k-algebra k' ⌈$\in \mathrm{Ob}\,\mathbb{G}_k$, cf. §1⌋ we look at the affine space k'^I. We would like to interpret k'^I as the set of k-alg homomorphisms from P_I to k' ⌈i.e. as $\mathrm{Hom}_{\mathbb{G}_k}(P_I, k')$⌋ (cf. §2),

$$k'^I \xrightarrow{\sim} \mathrm{Hom}_{k\text{-alg}}(P_I, k') = \mathbb{E}^I(k') , \tag{5.1}$$

where this bijection is functorial with respect to k. This functor,

$$\mathbb{E}^I : \mathbb{G}_k \to \mathrm{Sets} , \tag{5.2}$$

⌈defined by $k' \to \mathrm{Hom}_{\mathbb{G}_k}(P_I, k')$⌋ is called the standard affine space of

<u>type I over k</u>. The functor $\mathbb{E}^I$ is <u>represented</u>(*) by the k-algebra P_I.

The points of $\mathbb{E}^I(k')$ are called the k'-valued points of $\mathbb{E}^I$ (for every $k' \in \mathrm{Ob}\,\mathfrak{A}_k$).

Now if we have a bunch of polynomial equations

$$f_j(x) = 0 \qquad j \in J \tag{5.3}$$

indexed by an arbitrary index set J ($f_j(T_i) \in P_I$ (cf. §1), $\forall j \in J$) and we look at the set $V_S(k') \hookrightarrow \mathbb{E}^I(k')$ of all points of $\mathbb{E}^I(k')$ which satisfy S for a variable k' ($\in \mathrm{Ob}\,\mathfrak{A}_k$) we obtain a <u>subfunctor</u> $k' \mapsto V_S(k')$ of $k' \mapsto \mathbb{E}^I(k') = k'^I$.

The study of this bunch of equations S from the point of view of algebraic geometry is the study of this functor V_S.

Even in the classical case, when k is a field, for instance, the field $\mathbb{R}$ of the reals and the number of equations and indeterminates is finite, it may happen that the set $V_S(\mathbb{R})$ is empty, for instance: look at the case of the single equation in one variable $x^2 + 1 = 0$ or at the case $x^2 + y^2 = 0$ in the real plane. However <u>we do not consider that the "varieties" defined by either one of these equations are trivial</u>, because over suitable $\mathbb{R}$-algebras these equations have solutions: for instance, if we take $k' = \mathbb{C} \in \mathfrak{A}_{\mathbb{R}}$ (the field of complex numbers) we obtain the two solutions $\pm i$ in the first case or the whole continuum of complex-valued points of the imaginary circle in the second. So if we restricted ourselves to the real case we would have practically no information about the system S if we are not allowed to consider a variable

(*) The neophyte can realize how simple is this notion of <u>representable functor</u> in this concrete case needed here: $\mathbb{E}^I : \mathfrak{A}_k \to \mathrm{Sets}$; we have $k' \to \mathbb{E}^I(k') = \mathrm{Hom}_{\mathfrak{A}_k}(P_I, k')$, $\forall$ k-alg. morphism $k' \xrightarrow{u} k''$ induces $\mathbb{E}^I(u) : \mathrm{Hom}(P_I, k') \to \mathrm{Hom}(P_I, k'')$ in an obvious covariant way. It is clear that we did not use any particular property of the category $\mathfrak{A}_k$. Cf. §1 for the main definitions of representable functors.

$k' \in \mathrm{Ob}\,\mathcal{C}_k$; the geometric properties should be invariant under any base change (in the previous examples the property of $V_S(\mathbb{R})$ being empty is not something invariant under base change).

At the end we add some historical motivations to introduce $\mathbb{E}^I$, \
instead of restricting ourselves to k , in particular $\mathbb{R}, \mathbb{C}, \ldots$

Let us first introduce the standard affine linear space over k c
type I[1] to be a functor $\mathbb{E}^I$ from the category $\mathcal{C}_k$ of k-algebras t
the category of sets which associates to every $k' \in \mathrm{Ob}\,\mathcal{C}_k$ ($\Leftrightarrow$ k' is
k-algebra) the affine space k'^I , namely

(5.4) $$\mathbb{E}^I : \mathcal{C}_k \to \text{Sets}$$

(5.5) $$\mathbb{E}^I : k' \to k'^I \qquad \forall\, k' \in \mathrm{Ob}\,\mathcal{C}_k$$

Now, let us consider a morphism

(5.6) $$u : k' \to k''$$

in $\mathcal{C}_k$ i.e. a (k-alg)-homomorphism. cf. §2.

We have an induced map:

(5.7) $$\mathbb{E}^I(u) : k'^I \to k''^I \Leftrightarrow (x_i)_{i \in I} \mapsto (u(x_i))_{i \in I}$$

Thus $\mathbb{E}^I$ is actually a functor $\mathbb{E}^I : \mathcal{C}_k \to$ Sets.

THE FUNCTOR V_S: now, let us consider the system S of equation
(5.1) with coefficients in k , as a system of equations with coeffici

(1) $\mathbb{E}^I$ will appear soon as a particular case of the most general affine algebraic spaces; that is why we wrote linear, although affine linear may look redundant. This terminolo agrees with the customary consideration of affine algebraic varieties and affine spaces GROTHENDIECK refers to it often as the standard affine space of type I.

in k' and we look for the set of solutions

$$V_S(k') = \{x \in k'^I \mid f_j(x) = 0,\ \forall_j \in J\} \tag{5.8}$$

Then for every homomorphism (3.4) of k' into k'', and every point in k^I which satisfies S, the image in k''^I satisfies the same set of equations, i.e. $V_S(k')$ goes into $V_S(k'')$. Using functorial notations, we have: $k' \xrightarrow{u} k''$ in the category $\mathfrak{C}_k$ goes into

$$V_S(u) : V_S(k') \to V_S(k'') \tag{5.9}$$

in the category of sets.

In other words:

Our system of equations S defines a subfunctor V_S of $\mathbb{E}^I$, from $\mathfrak{C}_k$ to the category of sets:

$$k' \xrightarrow{V_S} \{x \in k'^I \mid f_j(x) = 0,\ \forall_j \in J\} \subset \mathbb{E}^I(k') = k'^I \tag{5.10}$$

i.e. $V_S(k')$ is the set of k'-valued solutions of S.

The set $V_S(k')$ is called a closed algebraic subset of the affine space $\mathbb{E}^I(k')$.(1)

[EXAMPLES AND HISTORICAL MOTIVATION. Besides the fact that the arithmetic problem (= to find $V_S(k)$) might be very difficult, we saw that in some cases $V_S(k)$ is empty, for instance in the case of the imaginary quadrics $\sum_{i=1} x_i^2 + 1$ in $\mathbb{R}^n$. To the inclusion $\mathbb{R} \hookrightarrow \mathbb{C}$ corresponds $\emptyset \hookrightarrow V_S(\mathbb{C})$. It is obvious that such transitions from the real to the complex domain are old and very frequent. Several problems both algebraic and arithmetic lead to the extension of the ground field k (cf. DIEUDONNÉ, Advances, I). Then the new problem arises of comparing points with coordinates in different extension fields k', k''. They cannot be compared at all if both are no subfields of a third field. A. WEIL introduced his universal domain Ω (a field of infinite transcendence degree over its prime field) assuming that all needed extensions are subfields of Ω. GROTHENDIECK does not need the ground field and rejects Ω. As a consequence he considers the different $V_S(k')$ for variable k' (where k' are k-algebras, after a suggestion of KÄHLER) and introduces instead the systematic consideration of the functor V_S, renouncing to the unique set $V_S(k)$ "algebraic variety" (fixed k) represented by S.

Geometrically the k-algebras play the role of "k-valued functions" over V_S. (The k-alg. homomorphism satisfied S as well as the "constant" points with coordinates in k). They were used in classical algebraic geometry under the name of "coordinate rings": $k' = k[\xi_1, \ldots, \xi_n]$

(1) We shall soon see a motivation for this terminology based on the so-called ZARISKI topology on affine spaces k^I (cf. Ch. I, §14).

of affine irreducible algebraic varieties in the particular case that k is a ground field, for instance $k = \mathbb{C}$ and $I = \{1,2,\dots,n\}$ finite. Then $k[\xi_1,\dots,\xi_n]$ is a finitely generated integral domain. $\xi_1,\xi_2,\dots,\xi_n$ are the coordinates of VAN DER WAERDEN "allgemeine Punkt". k' is isomorphic with the quotient ring of the polynomial ring $k[x_1,\dots,x_n]$ by a prime ideal $\mathfrak{J}$.

We shall see that in the general case of GROTHENDIECK every k' can be represented by $P_I/\mathfrak{J}$ where $\mathfrak{J}$ is an arbitrary (not necessarily prime) ideal of the polynomial ring P_I in an arbitrary set $(T_i)_{i \in I}$ of indeterminates. For I fixed, to different ideals of P_I correspond different *functors*. *If we want just k-valued points this is false*, for instance $x^2 + y^2 - 1 = 0$ and $(x^2 + y^2 - 1 = 0$ and $(x^2 + y^2 - 1^2)^m = 0$ define the same set complex-valued solutions for every $m \geq 1$, thus the principal ideals $(x^2 + y^2 - 1)^m$ $(m = 1,2,\dots)$ define all the same $V_s(\mathbb{C})$. Cf. next §.

6. INTRINSIC STUDY OF V_S AND EMBEDDING $V_S \hookrightarrow \mathbb{E}^I$.

SUMMARY. Our algebraic-geometric problem has been reduced to the study of the subfunctor V_S of $\mathbb{E}^I$ and we will consider two different questions: a) Intrinsic study of V_S, b) Study of embeddings $V_S \hookrightarrow \mathbb{E}^I$. These two points of view are parallel to distinguishing intrinsic properties of curves or surfaces independent of any embedding in 3-space (or n-space, $n \geq 3$) from those depending on the curve or surface *and* the embedding, in elementary differential geometry.

The subfunctor V_S can be viewed in two different ways: a) first as an intrinsic study of the functor V_S, (considered up to isomorphism), which is just the study of a functor from the category $\mathfrak{C}_k$ of k-algebras to the category of sets. Thus V_S and $V_{S'}$ (where S' is another system of equations possibly with different index sets I',J' defined over the same ground ring k) and b) second as the study of the particular properti of the embedding $V_S \hookrightarrow \mathbb{E}^I$.

EXAMPLE.(1) An example of viewpoint a) is found in the study of real or complex algebraic varieties lying in some affine space of finite dimens: N, independent of the embedding (in such a way that for $N \neq N'$ it makes sense to define an equivalence relation identifying certain pairs of affin varieties). For instance the reader can think of algebraic curves or surfaces embedded in real or complex three space. When we take into accou some particular embedding of the same varieties, we have an example of b).

(1) A classical example where one indeed uses outside points is the consideration of polar curves, tangent lines, normal lines,...etc...in the non intrinsic study of algebraic plane curves. The intrinsic study is *higher* and more sophisticated.

The new fact is the functorial approach; k' is variable! (instead of keeping $\mathbb{R}$, or $\mathbb{C}$ fixed as in elementary differential or classical algebrai geometry). The distinction between intrinsic and extrinsic properties is old

7. REDUCTION TO THE CASE OF S = IDEAL OF P_I .

SUMMARY. [Two systems S, S' of polynomial equations (of type (I,J), (I',J')) with coefficients in k are considered as equivalent, iff the two functors of solutions $V_S, V_{S'} : \mathcal{C}_k \to \text{Sets}$ are equivalent.[1]

Let us keep I fixed, or what is the same the "ambient space" $\mathbb{E}^I$. We can enlarge S
with all the finite linear combinations $\sum_{j\in J} g_j f_j$ generating an ideal $\mathfrak{J} = P_I \cdot S$ of P_I .
Then $V_S = V_{\mathfrak{J}}$, thus we can restrict ourselves to the study of functors of type $V_{\mathcal{J}}$ ($\mathcal{J}$ ideal
of P_I) . Moreover if $\mathcal{J} \neq \mathfrak{J}$ ($\mathcal{J}, \mathfrak{J}$ ideals of P_I) the functors $V_{\mathcal{J}}, V_{\mathfrak{J}}$ are not equivalent;[2]
in other words:

We shall see now that there is a (1-1)-correspondance between the subfunctors of $\mathbb{E}^I$
that can be described by sets of polynomial equations and the subfunctor defined by all the
ideals $\mathfrak{J}$ of P_I .]

Now, whether our point of view is arithmetic or geometric, a first observation coming to our minds is that we can in many ways enlarge the system of equations S without changing the solutions either in the ground ring k or in any arbitrary k-algebra k' . Namely if we take any linear combination $\sum_{j\in J} g_j f_j$ (with finitely many coefficients $g_j \in P_I$ different from zero) then any solution x of the set S is also a solution of $\sum g_j f_j = 0$.

So we can enlarge S in order to include all the finite linear $\sum_{j\in J} g_j f_j$. But this set of all such possible linear combinations is the ideal $\mathfrak{J} = \mathfrak{J}_S = P_I \cdot S$ generated by S . Thus we have for every k-algebra k' :

[1] I.e. for every $k' \in \mathrm{Ob}\,\mathcal{C}_k$, there exists a set equivalence: $h(k') : V_S(k') \to V_{S'}(k')$.

[2] This shows that even in the case that I is infinite and k is a commutative field, it is
not convenient to restrict ourselves to the case of S finite, since the most natural
assumption for S is that S is an ideal of P_I . But an ideal of P_I is rarely a finite
set. However in classical algebraic geometry when k is a field and I finite, we can
always assume S finite as a consequence of the basis theorem of HILBERT for ideals of P_I .
Even in this case the restriction of I being finite looks a priori artificial and tech-
nically disturbing.

(7.1) $\{x \in k'^I \mid f_j(x) = 0,\ \forall j \in J\} = \{x \in k'^I \mid f(x) = 0,\ \forall f \in \mathfrak{J}\}$.

In fact $V_{\mathcal{J}} \subset V_S$ because $S \subset \mathcal{J}$. Conversely $V_S \subset V_{\mathcal{J}}$ (a consequence of the previous remark).

Thus, we see that the two functors V_S and $V_{\mathfrak{J}}$ $(: \mathfrak{C}_k \to \text{Sets})$ are equivalent:

(7.2) $$V_{\mathcal{J}} = V_S$$

Since many different sets of equations S can define the same set of solutions for every $k' \in \mathrm{Ob}\,\mathfrak{C}_k$, choosing the ideal $\mathfrak{J}$ of the ring P_I in order to define the solution set $V_{\mathfrak{J}}(k')$ is the most natural choice that comes to mind! As we shall shortly see, this choice is a canonical choice. [Cf. footnote (2) in page 30.]

Let us elaborate on the correspondence between closed algebraic subspaces of the standard affine space $\mathbb{E}^I$ and ideals [of P_I]. To every ideal $\mathcal{J} \subset P_I$ we associate the subfunctor $V_{\mathcal{J}} \hookrightarrow \mathbb{E}^I$ [(1)]. We claim the ideal $\mathcal{J}$ can be reconstructed from the knowledge of $V_{\mathcal{J}}$. In fact one gets a (1-1)-correspondence

(7.3) $$\mathcal{J} \mapsto V_{\mathcal{J}}$$

between ideals $\mathcal{J} \subset P_I$ and subfunctors of $\mathbb{E}^I$ of the type previously described. How can we recover $\mathcal{J}$ from $V_{\mathcal{J}}$?

An element $f \in P_I$ belongs to $\mathcal{J}$ iff for every $k' \in \mathrm{Ob}\,\mathfrak{C}_k$ and for every k'-valued point x of $V_{\mathcal{J}}$ (i.e. $x \in V_{\mathcal{J}}(k')$, cf. §1) we have

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(1) A k'-valued point $u \in \mathrm{Hom}_{k\text{-alg}}(A,k')$ is mapped to a point of the subset $V_{\mathcal{J}}(k') \to \mathbb{E}^I$ $(u \mapsto (u(e_i))_{i\in I})$. $V_{\mathcal{J}}(k')$ is the set of all the points of $\mathbb{E}^I(k')$ satisfying the system of equations $f(x) = 0,\ \forall f \in \mathcal{J}$.

(7.4)
$$f(x) = 0$$

In fact, it is clear that if $f \in \mathcal{J}$ the equation (7.4) is satisfied. The converse property is true: any $f \in P_I$ such that (7.4) holds for any k'-valued point of $V_{\mathcal{J}}(k')$ ⌈for any choice of $k' \in \mathrm{Ob}\,\mathcal{C}_k$⌋ belongs to $\mathcal{J}$. In fact we can take $k' = P_I/\mathcal{J}$. Then $f(x) = 0$ for $x \in P_I/\mathcal{J}$ iff $f \in \mathcal{J}$ since $\mathcal{J}$ is the kernel of this canonical homomorphism $P_I \to A = P_I/\mathcal{J}$. Moreover <u>the map</u> $\mathcal{J} \mapsto V_{\mathcal{J}}$ <u>is order reversing</u>!

Let $\mathcal{J} \subset \mathcal{J}' \subset P_I$ be an inclusion between ideals of P_I. Then we have an inclusion in the opposite direction $V_{\mathcal{J}'} \subset V_{\mathcal{J}}$.

Conversely, the inclusion $V_{\mathcal{J}'} \subset V_{\mathcal{J}}$ implies $\mathcal{J} \subset \mathcal{J}'$ because of the previous characterization of $\mathcal{J}$. In other words:

<u>The map</u> $\mathcal{J} \to V_{\mathcal{J}}$ <u>is bijective because it is order preserving in both directions</u>.

Thus, we get a perfect dictionary between the language of ideals in P_I and the language of subfunctors $V_{\mathcal{J}} \hookrightarrow \mathbb{E}^I$ (⇔ "closed algebraic subspaces" of P_I). Therefore we can say, if we insist in studying affine algebraic spaces embedded in $\mathbb{E}^I$, this study is equivalent to the study of ideals in P_I. That is why "old-timers" said that affine algebraic geometry is just "the same" as ideal theory in P_I. ⌈A criticism of this <u>non-intrinsic point of view</u>, will be given in §8.⌋

HISTORICAL REMARKS. We pointed out already (cf. Summary of the course, page 6) that in classical algebraic geometry $P_I = k[T_1, T_2, \dots, T_n]$ $(n = \# I < \infty)$. If k is algebraically closed (for instance $k = \mathbb{C}$) the most interesting ideals are <u>the radical ideals</u> $\mathcal{J}$ (cf. footnote (5), page 5). $(\Leftrightarrow \mathcal{J} = \sqrt{\mathcal{J}})$. In fact, because of HILBERT's Nullstellensatz, $A = k[T_1, T_2, \dots, T_n]/\mathcal{J}$ is the ring of polynomial functions $f: M \to k$ of some algebraic

(1) A k'-valued point $u \in \mathrm{Hom}_{k\text{-alg}}(A, k')$ is mapped to a point of the subset $V_{\mathcal{J}}(k') \to \mathbb{E}^I$ $(u \to (u(a_i))_{i \in I})$. $V_{\mathcal{J}}(k')$ is the set of all the points of $\mathbb{E}^I(k')$ satisfying the system of equations $f(x) = 0$, $\forall f \in \mathcal{J}$.

variety M (non necessarily irreducible[1]) iff $\mathcal{J} = \sqrt{\mathcal{J}}$. On the contrary if $\sqrt{\mathcal{J}}$ is stric larger than $\mathcal{J}$ A has nilpotent elements $f \neq 0$ ($\Leftrightarrow \exists$ integer $m > 1$ such that $f^m = 0$) It is clear that f cannot represent any function on the algebraic variety M represented $\mathcal{J}$.[2] In spite of this "forbidding" situation GRÖBNER made an attempt[3] to establish a bi $\mathcal{J} - V_{\mathcal{J}}$ between arbitrary ideals and affine algebraic varieties of k^n disregarding the fa that $\mathcal{J}$ and $\sqrt{\mathcal{J}}$ define the same variety (as locus of k-valued points) (the only geometric points allowed in those times). He was criticized for being "tautological" (cf. for instan SAMUEL[4]). Really the tautology disappears completely only with the introduction of the structure sheaf $\mathcal{O}_X$ of the spectrum $X = \operatorname{Spec} A$ (cf. Ch. III). Then A becomes the of global sections of $\mathcal{O}_X$ (and $\mathcal{O}_X$ can be defined always, whether or not there are non tr nilpotent elements).

8. THE CATEGORY OF AFFINE ALGEBRAIC SPACES OVER k . THE "ABSOLUTE" CASI $k = \mathbb{Z}$.

GROTHENDIECK's SUMMARY:

Talking about old-timers, [cf. Summary of §6].......from the intrinsic point of view we are working through now, we are not really interested in the embedding of algebraic spaces into particular standard affine spaces. Thus the ideal $\mathcal{J}$ is not so interesting in itself. What is interesting is the quotient algebra $A = P_I/\mathcal{J}$[5] since this quotient will represent the functor in which we are interested. Precisely, we shall see the functors $\mathbb{G}_k \to \mathrm{Sets}$ isomorphic to functors V_S characterized as representable functors[6] Σ_A , i.e. those for which there exists a k-algebra A , the representing object,[7] such that (8.2) holds.

In terms of the ideals of P_I , we define A as $P_I/\mathcal{J}$ i.e. we have the equivalence of functors $\mathbb{G}_k \to \mathrm{Sets}$: $V_S = V_{\mathcal{J}} = \Sigma_A$ with $\mathcal{J} = P_I \cdot S$, $A = P_I/\mathcal{J}$. Σ_A has the advantage of being independent of any particular embedding in any standard affine space. Conversely any k-algebr A defines a Σ_A which is equivalent to some V_S , since every A is a quotient $P_I/\mathcal{J}$ for suitable choice of I and $\mathcal{J}$. The functor Σ_A ($A \in \mathrm{Ob}\,\mathbb{G}_k$) is called the affine algebraic space over k represented by A .

The affine algebraic spaces over k form a category anti-equivalent[8] to the category $\mathbb{G}_k$ of k-algebras.

For $k = \mathbb{Z}$ we obtain the category of "absolute" affine algebraic spaces anti-equivalent to the category $\mathbb{G} = \mathbb{G}_{\mathbb{Z}}$ of commutative rings with unit.

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(1) M is irreducible iff $\mathcal{J}$ ($= \sqrt{\mathcal{J}}$) is prime.

(2) Any $\mathcal{J} \subset k[T_1,\dots,T_n]$ represents a variety $M_{\mathcal{J}}$ but different ideals might represent t same M . However there is a biggest one representing $M_{\mathcal{J}}$, precisely $\sqrt{\mathcal{J}}$. §127, pag

(3) GRÖBNER, Moderne algebraische Geometrie (Die idealtheoretischen Grundlagen), Springer Verlag, Wien und Innsbruck, 1949.

(4) SAMUEL, Méthodes d'algèbre abstraite en Géométrie algébrique, Ergebnisse der Math, Band 2nd edition, 1967: "Quand à W. GRÖBNER..., tandis que ses "Algebraische Mannigfaltigke ont l'air d'être à peu près des idéaux de polynômes. page 128.

(5) [Remember the geometric meaning of $P_I/\mathcal{J}$ in the classical case recalled at the end of

(6) cf. §1. We have already found the representable functors in the particular case $k =$ P_I represents the standard affine space $\mathbb{E}^I$.

(7) A is determined up to isomorphism (cf. Ch. II, §1).

(8) I.e. equivalent to the opposite category $\mathbb{G}^\circ$ of $\mathbb{G}$.

To say that the functor $V_{\mathcal{J}}$ ($\mathcal{J}$ ideal of P_I) is represented by $A = P_I/\mathcal{J}$ is equivalent to express the following universal property of the canonical homomorphism $P_I \overset{p}{\to} A$:

Let $x: P_I \to k'$ be a k'-valued point of P_I ($k' \in \mathrm{Ob}\,\mathfrak{A}_k$) (cf. §2). Then $f(x) = 0$, for every $f \in \mathcal{J}$ if and only if x factors uniquely through p , i.e. if there is a commutative diagram

(8.1)
$$\begin{tikzcd} P_I \arrow[r, "x"] \arrow[d, "p"'] & k' \\ A \arrow[ur, "x_A"'] & \end{tikzcd}$$

i.e. $x = x_A \circ p$ with x_A uniquely determined by x .

The map $x \mapsto x_A$ defines a bijection of the set $V_{\mathcal{J}}(k')$ of k'-valued points of $\mathbb{E}^I$ vanishing at $\mathcal{J}$ and the set $\mathrm{Hom}_{\mathfrak{A}_k}(A,k')$; q.e.d. in other words:

The functor $V_{\mathcal{J}}: \mathfrak{A}_k \to \mathrm{Sets}$ is representable by the k-algebra $A = P_I/\mathcal{J}$. The covariant representable functor $\mathfrak{X}_A: \mathfrak{A}_k \to \mathrm{Sets}$ defined by

(8.2) $$k' \mapsto \mathfrak{X}_A(k') = \mathrm{Hom}_{\mathfrak{A}_k}(A,k') \qquad \forall k' \in \mathrm{Ob}\,\mathfrak{A}_k$$

is called the affine algebraic space over k represented by the k-algebra A .

The previous construction eliminates $\mathbb{E}^I$: apparently there is a restriction on A since A is generated by the images $a_i = p(T_i)$ ($\forall i \in I$) . Actually there is not such a restriction. We can always choose a system of generators for A (as a k-algebra) (for instance we can take the whole underlying set of A). Then we can reverse the procedure. Following GROTHENDIECK's own words: We shall see that to give such an embedding $\mathfrak{X}_A \hookrightarrow \mathbb{E}^I$ as a morphism from $\mathfrak{X}_A$ to $\mathbb{E}^I$ we must consider as a morphism in the opposite direction between the representing algebras: $P_I \to A$. However this is equivalent to giving the images a_i of the generators T_i ($\forall i \in I$) of which means that the homomorphism $P_I \to A$ uniquely determined by T_i ' $(i \in I)$ is an epimorphism, which in turn is equivalent to the fact th quotient of P_I by some ideal $\mathcal{J}$.

We solve completely the problem of characterizing the functors $V_S : \mathfrak{A}_k \to \mathrm{Sets}$ independently of any affine embedding:

A functor $V_S : \mathfrak{A}_k \to \mathrm{Sets}$ represents the solutions of some sys of polynomial equations with coefficients in k if and only if V_S representable, i.e. iff V_S is equivalent to $\mathfrak{h}_A$ for some $A \in \mathrm{Ob}\,\mathfrak{A}$

Conversely: for every k-algebra $A^{(*)} \in \mathrm{Ob}\,\mathfrak{A}_k$ the representab functor $\mathfrak{h}_A$ is equivalent to some V_S. $\mathfrak{h}_A$ is equivalent to $\mathfrak{h}_B$ $(A, B \in \mathrm{Ob}\,\mathfrak{A}_k)$ iff A and B are isomorphic.

The representable functors $\mathfrak{h}_A (A \in \mathrm{Ob}\,\mathfrak{A}_k)$ are the objects of a c Aff_k equivalent to the opposite category $\mathfrak{A}^{\circ}$.

[The morphisms in Aff_k are defined in a standard categorical wa as functor morphisms i.e. natural transformations, i.e. they are the induced maps f^* obtained from morphisms $B \xrightarrow{f} A$ in $\mathfrak{A}_k$:]

$$\mathrm{Hom}_{k\text{-alg}}(A, k') \to \mathrm{Hom}_{k\text{-alg}}(B, k')$$

by composition with $f : B \xrightarrow{f} A \xrightarrow{u} k' : u \to u \circ f$, $\forall u \in \mathfrak{h}_A(k')$ $(\forall k' \in \mathrm{O}$

In GROTHENDIECK's own words:

(*) In classical algebraic geometry k was a field and A was finitely generated of $k[\xi_1, \dots, \xi_n]$ is an integral domain we get back the "allgemeine Punkt" of an irre algebraic variety (over k) embedded in k^n. If $k[\xi_1, \dots, \xi_n]$ is a reduced alg the field k still we can give a classical interpretation (cf. Summary of the co In the general case there is no classical interpretation at all...

The reason why I was not supposed to be finite appears clear now. We want the category of k-algebras, not just the finitely generated ones. Finite dimensional af spaces appear just when I is finite.

(**) Cf. Ch. II, §1 for further categorical elaborations. It is clear that we did not u specific property of the category $\mathfrak{A}_k$.

If $\mathfrak{X}$, $\mathfrak{X}'$ are affine algebraic spaces over k represented by k-algebras A, A' the morphisms $\mathfrak{X} \to \mathfrak{X}'$ in the category Aff_k correspond bijectively (in the opposite direction) to the homomorphisms of k-algebras:

$$\mathrm{Hom}_{Aff_k}(\mathfrak{X},\mathfrak{X}') \cong \mathrm{Hom}_{k\text{-}alg}(A',A)$$

This is just the usual sorites of representable functors!: Arrows between the functors correspond to arrows between the objects. (Cf. last footnote).

REMARK. These facts look too formal, or tautological. The really interesting facts come from the association (in a functorial manner) to every A ($\Leftrightarrow$ every $\mathfrak{X}_A$) of a certain geometric object: the affine scheme defined by A: (a certain locally ringed space $(\mathrm{Spec}\, A, \tilde{A})$, (cf. Ch. III).

For the time being we have the equivalence of categories: $Aff_k \cong \mathcal{C}_k^{\circ}$. For $k = \mathbb{Z}$ we obtain the category $Aff = Aff_{\mathbb{Z}}$ of "absolute" affine algebraic spaces, anti-equivalent to the category $\mathcal{C} = \mathcal{C}_{\mathbb{Z}}$ of commutative rings with unit. We feel this is a geometric theory due to the geometric meaning of all the constructions involved:

If $\mathfrak{X}' \to \mathfrak{X}$ is a morphism in Aff_k <u>every k'-valued point in</u> $\mathfrak{X}'$ goes to a k'-valued point of $\mathfrak{X}$ (cf. §2) (just by pull-back of homomorphisms). In particular this is true for $k' = k$. $\mathfrak{X}' \overset{f}{\hookrightarrow} \mathfrak{X}$ is injective iff the corresponding $f^*: A \to A'$ is surjective, thus we come back to the initial remarks that f^* represents a restriction from $\mathfrak{X}$ to $\mathfrak{X}'$. In particular we can choose a $\mathfrak{X} \hookrightarrow \mathbb{E}^I$ as before. In the classical case one is particularly interested in the case that k' is a reduced algebra $(\mathrm{Nil}\, k' = 0)$ or it is an integral domain, or in particular a field. Besides $\mathfrak{X}(k')$ is a closed algebraic subset of some $\mathbb{E}^I$ (if we want to use affine embeddings again, although we tried often to avoid them). In the next sections we shall study these particular kind of "points" of $\mathfrak{X}$, in particular when the ring A representing $\mathfrak{X}$ belongs to one of these special types. Then if k' is a field the "geometric points" (cf. next §9) will give us back the intuition of the classical case.

PART II

RESTRICTION TO PARTICULAR TYPES OF k-ALGEBRAS
($k' = k$ fixed, $k' = \mathbb{C}$, k' a field, k' reduced,...)

9. [SUMMARY. We come back to the system S of polynomial equations $f_j(T_i) = 0$ ($i \in I$, $j \in J$), with coefficients in a ground ring k (cf. §1), but instead of taking arbitrary test algebras $k' \in \mathrm{Ob}\,\mathfrak{A}_k$, we shall restrict k' to certain subcategories of $\mathfrak{A}_k$, which have a particular interest for some reason, geometric, historical, important for the foundations of algebraic geometry. Actually these cases help to understand GROTHENDIECK's simplifications. The main cases are:

1) $k' = k$ fixed, i.e we fix a subcategory Fix k with only one object i.e we allow only k-valued points, It is the <u>arithmetic</u> point of view again, cf. §5 -

2) $k' \in \mathrm{Ob}\,\mathfrak{F}_k$ ($\mathfrak{F}_k$ full subcategory of $\mathfrak{A}_k$ whose objects are <u>fields</u>)

3) The k-algebras k' are <u>reduced</u> (i.e. k' contains no nilpotent elements $\neq 0$). We can still reduce, as before the case of an arbitrary S to the case of an ideal $\mathfrak{J}$ of P_I but the relation between the $\mathfrak{J}$ and $V_{\mathfrak{J}}$ is not quite so simple as in the general case.

In the case 1) if k is the field of complex numbers V_S, becomes the complex algebraic variety $V_-(\mathbb{C}) \subset \mathbb{C}^I$. In the case of finite I, this was one of the main subjects of study of the pioneers of the XIXth century: RIEMANN, Max NOETHER and then the Italians, etc. The study of irreducible (affine or projective) algebraic varieties was considered as THE object of classical algebraic variety. Let us follow GROTHENDIECK's own words:]

...Now let us see what happens if instead of taking arbitrary algebr k' and k we took only certain types of algebras, for instance in the first naive point of view we restricted ourselves to the case $k' = k$ (k' fixed) (cf. case 1) ; for instance initially we can assume $k' = k$ (the field of real numbers) and then the next step would be the algebraic geometric over the field $\mathbb{C}$ of complex numbers (and we shall not move it

(1) The affine $V_S(\mathbb{C})$ were considered for local problems. For global problems, classical algebraic varieties were always the projective ones.

any more)[1]; this is the classical case, cf. §10, or also we can take only algebras k' over k which are just fields or we can consider only <u>reduced algebras</u> k'[2]

10. FIELD VALUED POINTS:

SUMMARY: We shall restrict the functor $V_{\mathfrak{J}}$ to the full subcategory $\mathfrak{F}_k$ of $\mathfrak{A}_k$ consisting of fields which are k-algebras. Precisely for every ideal $\mathfrak{J}$ of P_I we define $V'_{\mathfrak{J}}: \mathfrak{F}_k \to \text{Sets}$ as the restriction

$$V'_{\mathfrak{J}} = V_{\mathfrak{J}}|\mathfrak{F}_k$$

Then <u>the bijective map</u> $\mathfrak{J} \mapsto V_{\mathfrak{J}}$ <u>is lost</u>; what we get is the following:

<u>Two ideals</u> $\mathfrak{J}, \mathfrak{J}'$ <u>of</u> P_I <u>define the same functor iff</u> $\mathfrak{J}$ <u>and</u> $\mathfrak{J}'$ <u>have the same radical</u>:

$$V'_{\mathfrak{J}} = V'_{\mathfrak{J}'} \iff \sqrt{\mathfrak{J}} = \sqrt{\mathfrak{J}'}$$

As a consequence <u>there is a bijective map between the functors V' and the radical ideals</u>:

$$\mathfrak{J} = \sqrt{\mathfrak{J}} \mapsto V'_{\mathfrak{J}} (= V'_{\sqrt{\mathfrak{J}}})$$

The proof requires the lemma $\bigcap_{\mathfrak{p} \in \operatorname{Spec} A} \mathfrak{p} = \operatorname{Nil} A$ equating two definitions of the nilradical of A.

The proof of the bijection between the sets of radical ideals of P_I and functors $V'_{\mathfrak{J}}$ can be extended to other cases, for instance when the k' are <u>reduced</u> (or, in particular when the k' are integral domains). Anyhow the statement for fields is particularly significant because of the importance of the <u>geometric points</u>, cf. next §11.

...So, if we look at points of the affine space k'^I with k' a field $k' \in \operatorname{Ob} \mathfrak{F}_k$, $\mathfrak{F}_k$ full subcategory of $\mathfrak{A}_k$), we restrict the functor $V_{\mathfrak{J}}: \mathfrak{A}_k \to \text{Sets}$ to the functor

$$V'_{\mathfrak{J}} = V_{\mathfrak{J}}|\mathfrak{F}_k : \mathfrak{F}_k \to \text{Sets} \tag{10.1}$$

going from the category $\mathfrak{F}_k$ of fields k which are k-algebras to the category of sets. So we shall look at points of an affine space $\mathbb{E}^I(k')$

[1] [Cf. the classical case recalled in the Summary

[2] I.e. without nilpotent elements $\neq 0$, cf. footnote (1), page 9

with coordinates in a field $k' \left[\in \mathrm{Ob}\,\mathfrak{F}_k\right]$; if $f \in P_I$ and we assume $(f(x))^m = 0$ for some positive integer m $(x \in k'^I)$ we have $f(x) = 0$. This implies that <u>if a power of a polynomial vanishes at all the points o</u> $V'_{\mathcal{J}}(k')$ $(k' \in \mathrm{Ob}\,\mathfrak{F}_k)$ <u>then f itself vanishes identically on</u> $V'_{\mathcal{J}'}(k')$. This means that if we replace the ideal $\mathcal{J}$ by its radical $\sqrt{\mathcal{J}}$[1] $(\sqrt{\mathcal{J}} \supset$ then we have

$$V'_{\mathcal{J}} = V'_{\sqrt{\mathcal{J}}}{}^{(2)} \tag{10.2}$$

Taking the ideal $\sqrt{\mathcal{J}}$ associated to $\mathcal{J}$ just means that if I look at the quotient ring $P_I/\mathcal{J}$, then $\sqrt{\mathcal{J}}$ is the inverse image of the nil-radical of $P_I/\mathcal{J}$. So what we did in terms of these quotient rings is divide out $P_I/\mathcal{J}$ by its nilradical $\mathrm{Nil}(P_I/\mathcal{J})$, in such a way that the quotient $(P_I/\mathcal{J})/\mathrm{Nil}(P_I/\mathcal{J})$ is reduced:

$$P_I/\sqrt{\mathcal{J}} \simeq (P_I/\mathcal{J})/\mathrm{Nil}(P_I/\mathcal{J}) \tag{10.3}$$

Now let us see what that means! We can check easily that $V'_{\mathcal{J}} \subset V'_{\mathcal{J}'}$ if $\sqrt{\mathcal{J}'} \subset \sqrt{\mathcal{J}}$. Then if we take this for granted we obtain a bijective, orc reversing correspondence between the set of radical ideals of P_I and subfunctors: $V'_{\mathcal{J}}: \mathfrak{F}_k \to \mathrm{Sets}$ coming from bunches of polynomial equatio In order to get this relation we have to see how to recover $\sqrt{\mathcal{J}}$ in te of $V'_{\mathcal{J}}$. Now let us express this fact in terms of commutative algebra Let us look therefore at the quotient ring $A = P_I/\mathcal{J}$. Let us recall

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(1) Cf. footnote (2) page 9.

(2) In other words, we are allowed now after restricting $\mathcal{C}_k$ to $\mathfrak{F}_k$ not only to const the ideal $\mathcal{J}$ of all the k-linear combinations of polynomials of S. We can go furt here, we can also consider the polynomial f such that a suitable power f^m $(m \in \mathbb{Z}^+$ an equation also!

the points u of $V'_{\mathcal{J}}$ with values in k' are just the homomorphisms:

(10.4) $$u: A \to k' \qquad (\forall\, k' \in \mathrm{Ob}\, \mathfrak{F}_k)$$

Then the property I stated can be rephrased as follows:

An element f of P_I has zero image by any homomorphism (10.4). Therefore what we are looking for just means that the intersection of all the kernels of homomorphisms u of type (10.4) is just $\sqrt{\mathcal{J}}$. Now this statement is equivalent to a well-known lemma of commutative algebra.[1] The kernels of homomorphisms of type (10.4) are precisely the prime ideals of A[2]. In other words: the mentioned lemma expresses that:

<u>The intersection of all the prime ideals of A is equal to the ideal of nilpotent elements of A</u> (= the Nilradical of A):

$$\bigcap_{\mathfrak{p} \in \mathrm{Spec}\, A} \mathfrak{p} = \mathrm{Nil}\, A$$

It is clear that if f is nilpotent, $f \in \mathfrak{p}$, $\forall \mathfrak{p} \in \mathrm{Spec}\, A$. The opposite implication is less clear. It requires the axiom of choice or ZORN's Lemma.

It can be seen as follows: if $f \in A$ is not nilpotent the localization with respect to the multiplicative set $S = \{f^n \mid n \geq 0\}$ (cf. Ch. III, § 8) is a non zero ring $S^{-1}A$, thus there exists a maximal ideal $\mathfrak{m} \in S^{-1}A$ (KRULL's Th.) and the inverse image $\mathfrak{p} = p^{-1}\mathfrak{m} \in \mathrm{Spec}\, A$ by the canonical homomorphism $p: A \to S^{-1}A$ is a prime ideal such that $f \notin \mathfrak{p}$.

(1) If f is an element of the ring $A \in \mathrm{Ob}\,\mathcal{G}$ such that $u(f) = 0$ for every homomorphism $u: A \to k$ $(k \in \mathfrak{F}_k)$ then f is nilpotent. This is a well-known result of Commutative algebra that, like many others of this type, can be deduced formally from KRULL's theorem: If $A \neq 0$ then A contains at least one maximal ideal.

(2) An ideal $\mathfrak{p}$ of A ($\in \mathrm{Ob}\,\mathcal{G}$) is <u>prime</u> iff $A/\mathfrak{p}$ <u>is an integral domain</u> cf. footnote (4) in page 9. In particular A is never prime because $A/A = 0$ is not an integral domain. The zero ideal of A is prime iff A is an integral domain.

11. THE CLASSICAL CASE: k FIXED ALGEBRAICALLY CLOSED.

SUMMARY: ...for instance, let us assume $k'(\in \mathrm{Ob}\,\mathcal{J}_k)$ to be fixed, equal to k . Then we obtain some nice results in the case that k is algebraically closed and the set $I = \{T_1, T_2, \ldots, T_n\}$ of indeterminates of our bunch of equations is finite.

Let us call $V''_{\mathcal{J}}$ the subset of k^I where all the polynomials of the idea $\mathcal{J}$ of P_I vanish (k-valued solutions of S!). Then it is still true tha $V''_{\mathcal{J}}$ depends only on $\sqrt{\mathcal{J}}$

(11.1) $V''_{\mathcal{J}} = V''_{\sqrt{\mathcal{J}}}$ $\quad \mathcal{J} \subset k[T_1, T_2, \ldots, T_n]$ (k alg. closed, $n < \infty$) .

Therefore it is true that if $\mathcal{J}, \mathcal{J}'$ are two ideals of $k[T_1, \ldots, T_n]$ which have the same radical the corresponding $V''_{\mathcal{J}}, V''_{\mathcal{J}'}$ are equal:

(11.2)
$$\sqrt{\mathcal{J}} = \sqrt{\mathcal{J}'} \Rightarrow V''_{\mathcal{J}} = V''_{\mathcal{J}'} \subset k^n$$

This is true because $\sqrt{\mathcal{J}}$ can be reconstructed from the knowledge c the algebraic closed subset $V''_{\mathcal{J}}$ of k^n . This is (essentially) the so-c HILBERT's Nullstellensatz. In other words, we can reconstruct (in terms $V''_{\mathcal{J}}$) , not the ideal $\mathcal{J}$ but its radical ideal $\sqrt{\mathcal{J}}$, by showing that $\sqrt{\mathcal{J}}$ the ideal of all the polynomials of $k[T_1, T_2, \ldots, T_n]$ vanishing on the se $V''_{\mathcal{J}}$ Classically $V''_{\mathcal{J}}$ was called the affine algebraic variety defined by the ideal $\mathcal{J}$ of $k[T_1, T_2, \ldots, T_n]$ in the affine space k^n .

We have, in other words, the following classical situation : Any ideal $\mathcal{J}$ of $k[T_1, \ldots, T_n]$ defines an affine algebraic variety $V''_{\mathcal{J}}(k) =$ (k fixed). The ideal of polynomials vanishing at every point of $V''_{\mathcal{J}}$ i precisely the radical $\sqrt{\mathcal{J}}$ of $\mathcal{J}$.

This is the classical version of HILBERT's Nullstellensatz. So we shall see what that means in an intrinsic language :

Let $\mathcal{J}$ an ideal of $P_I = k[T_1,T_2,\ldots,T_n]$. The points of k^n can again be identified with k-algebra homomorphisms $P_I \overset{u}{\to} k$ (cf. §2) and those satisfying the bunch of equations $\{f(x) = 0,\ \forall f \in \mathcal{J}\}$, (have the property $\ker u \supset \mathcal{J}$) factors through the quotient $A = P_I/\mathcal{J}$ making commutative the diagram:

$$\begin{tikzcd} P_I \arrow[r, "u"] \arrow[d, "p"'] & k \\ A \arrow[ur, dashed] & \end{tikzcd}$$

The statement is that if a polynomial $f \in P_I$ belongs to the intersection of all the kernels, $\ker u$, of algebra homomorphisms $u\colon P_I \to k$ vanishing at $V^n_{\mathcal{J}}$ ($\ker u \supset \mathcal{J}$) then a certain power f^m ($m \geq 1$) belongs to $\mathcal{J}$, i.e. <u>the image of</u> f <u>in</u> A <u>is nilpotent</u>. Since A is a finitely generated k-algebra (k alg. closed) the Nullstellensatz says that:

$$\operatorname{Nil} A = \{f \in A,\ f^m = 0 \text{ for some } m \in \mathbb{Z}^*\} = \bigcap_{u \in \operatorname{Hom}_{k\text{-alg}}(P_I;k) \text{ s.t. } \ker u \supseteq \mathcal{J}} \ker u$$

This statement can be decomposed into two (both regarding a finitely generated k-algebra A, with k algebraically closed field):

$$\text{Nullstellensatz} = \begin{cases} \text{a) If } A \text{ is a field, } A \cong k \\ \text{b) } \operatorname{Nil} A = \bigcap_{m \in \operatorname{Max} A} m \quad (1) \end{cases}$$

a) is not tautological because our assumptions <u>do not imply that A is finitely generated as a k-module.</u> This ingredient of the Nullstellensatz shows us that the hypothesis that k is algebraically closed <u>is essential</u> in order that a) remains true. Otherwise a) would be false for any non trivial finite field extension of k.

The property b) expresses the fact that $\operatorname{Nil} A$ which is known to be

(1) $\operatorname{Max} A = \{m \mid m$ maximal ideal of $A\}$ is called the <u>maximal spectrum of A</u>.

the intersection of all prime ideals of A is also [because of our strong assumptions on A] the intersection of something that can be imagined to be larger. b) is a very special property of algebras finitely generated over fields; it tends to be "extremely false" in general, for instance for local rings where there is just one maximal ideal.

The implication $\Rightarrow$ is clear. a) comes from the fact that if A is a field and $u: A \to k$ is a k-alg. homomorphism A decomposes in $k + \ker u$ but $\ker u = 0$ thus $A \simeq k$. b) is a consequence of the fact that the maximal ideals m of A are characterized by the condition that A/m is a field. But A/m is a finitely generated k-algebra and since k is algebraically closed $A/m \simeq k$. Therefore the maximal spectrum Max A is mapped bijectively in the set $\mathrm{Hom}_{k\text{-alg}}(A,k)$.

$$\mathrm{Max}\, A \xrightarrow{\sim} \mathrm{Hom}_{k\text{-alg}}(A,k)$$

Conversely if a) and b) are true the intersection of all the maximal ideals of A is the same as the intersection of kernels of k-alg homomorphisms of A into k and therefore the Nullstellensatz is verified.

REMARK. We noticed already that a) cannot be generalized to fields which are not algebraically closed, whereas <u>b) is true for every field k</u>. i.e. the intersection of the maximal ideals of A (finitely generated over k, k not necessarily algebraically closed field) is equal to the nilradical of A. This is not difficult to deduce from the case of k algebraically closed. a) can be replaced by a')

a') If A (finitely generated k-algebra) is a field then A is a finite algebraic field extension of k, i.e. A is also finitely generat as a k-module. a') can be restated also as follows:

If f belongs to the intersection of all the kernels of k-alg homomorphisms from A to a finitely generated field extension then <u>f is nilpotent</u>.

For proofs of the Nullstellensatz m we refer to ZARISKI-SAMUEL or BOURBAKI's Commutative Algebra.

12. EQUIVALENCE CLASSES OF POINTS OF $\mathfrak{X}$. GEOMETRIC POINTS.

To begin with let us consider a k algebra A, and let $\mathfrak{X}$ be the affine algebraic space represented by A, i.e. the representable functor $\mathfrak{X} = V_A$ assigns to every k-algebra k' the <u>affine algebraic set</u> $\mathrm{Hom}_{k\text{-alg}}(A,k')$ We are going to introduce an equivalence relation between k'-valued points $P\colon A \to k'$, for variable k'.

A k'-valued point, i.e. a k-alg. homomorphism

$$P\colon A \to k'$$

is not necessarily surjective. It defines a surjective map $P_S\colon A \to P(A)$, where $P(A)$ is a sub k-algebra of k', which might be strictly smaller. If P_S is surjective, then $P_S' \cong A/\ker P_S$, and it is possible to identify P_S with the canonical map $a \mapsto a + \ker P_S$ which is defined just by the kernel. The equivalence relation that we have in mind is defined by $P \sim P' \Leftrightarrow \ker P = \ker P'$, or what is the same $P \sim P'$ iff the correspondent surjective points $P_S \sim P_S'$ are equivalent. In other words <u>the set of these equivalence classes of points of A corresponds bijectively to the set of all ideals of A</u>, or what is the same with all homomorphic images $A/\mathfrak{a}$ and their canonical surjective homomorphisms $P: A \to A$ We are particularly interested in the so-called geometric points, i.e. those with values in k-algebra k' which is a field. Then P is geome iff $P(A) \subset P$ is an integral domain $\Leftrightarrow$ ker P_S = ker P is a prime id $\mathfrak{p}$. Conversely: for every prime ideal $\mathfrak{p} \in A$, the canonical projectio $A \to A/\mathfrak{p}$ defines an equivalence class of geometric points. Among the geometric points defined by $\mathfrak{p}$ there is a "minimal" one (cf. the field c fractions $K(\mathfrak{p})$ of the integral domain $A/\mathfrak{p}$). $A/\mathfrak{p}$ is itself a field if $\mathfrak{p}$ is a maximal ideal of A.

13. CRITICISMS ON NILPOTENT ELEMENTS:

GROTHENDIECK's SPEECH:

...So, we see that in certain problems, instead of taking all the k-algebras k' as "test-algebras" (in which we take coordinates of points of algebraic spaces) we took certain types, either fields, or integral domains or reduced rings, or fields which are finite extensions of k or if k is algebraically closed we took k' = fixed = k. The price for doing so is that we will no longer be able to distinguish (in geometric terms) between an ideal $\mathcal{J}$ of P_I and its radical $\sqrt{\mathcal{J}}$. In other words [if we reject again affine embeddings] and we think in terms of functors, intrinsically viewed, we look at functors $\mathfrak{X}_A$ (with $A = P_I/\mathcal{J}$) represented by A. Then, because of our restrictions on the k' we cannot distinguish A from its quotient $A/\mathrm{Nil}\,A$ ($\mathrm{Nil}(A/\mathrm{Nil}\,A) = 0$). So, the fact of working only with fields or with some types of fields k' implies technically that we work only with reduced rings A [as representing objects in the $\mathfrak{X}_A$]. This has been done for a very long time, until ten or fifteen years ago! Of course, people working in Commutative algebra look at rings which were not reduced [for instance $\mathbb{Z}/p^n\mathbb{Z}$, $P_I/\mathfrak{r}^n P_I$ ...] but, in Geometry in a way one refused to consider geometric interpretations of rings unless they were reduced! A very unpleasant situation! KÄHLER was the first one to systematically build up a context in which to associate geometric objects to rings which may have nilpotent elements. DIEUDONNE and I continued in the same direction. So it is now quite evident from great number of developments that have been done in the last fifteen years that, in fact, in Algebraic Geometry, working only with reduced rings does not give us all the results that one would expect. In many questions the nilpotent element are the crux of the matter. In many problems, for instance in infinitesimal questions, the nilpotent elements play a considerable rôle.

Therefore I think we should retain as a general principle that we shall not restrict our algebras k' over k in any way whatever, certainly not assuming that they are reduced algebras, i.e. the k-algebra A will never be confused with $A/\mathrm{Nil}\,A$, in order to eliminate nilpotent elements, since the nilpotent elements carry very valuable information concerning the ring A.

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(1) Cf. footnote (1) in page 41. According to an old-fashioned definition ($\mathfrak{p}$ prime $\Rightarrow a \notin \mathfrak{p}$, $b \notin \mathfrak{p} \Rightarrow ab \notin \mathfrak{p}$) A was considered as prime. It is important to keep in mind that we follow GROTHENDIECK's theory, because of the importance of the set

$$\mathrm{Spec}\,A = [\mathfrak{p} \mid \mathfrak{p} \text{ prime ideal of } A] \qquad \text{(the spectrum of } A)$$

(cf. CH. III). Thus $\mathrm{Spec}\,A = \emptyset \Leftrightarrow A = 0$ (KRULL's theorem).

I would like to comment why classical algebraic geometers did not consider nilpotent elements. If k is algebraically closed ($k = \mathbb{C}$, classically) the affine algebraic space (= affine variety considered at that time was identified with the maximal spectrum Max A. Then, for every $f \in A$ we can associate the function $\tilde{f}$: $\tilde{f}: m \mapsto \tilde{f}(m)$ = image of f in the residue field of $A_m (= k)$ and this function can be considered as a "polynomial function":

The map $f \to \tilde{f}$ is a ring isomorphism, because $\tilde{f}$ is identically zero iff $f \in m$ for every $m \in \text{Max } A \Rightarrow f$ is nilpotent $\Rightarrow f = 0$ (because A is reduced (cf. §11, page 42).

<u>Classically</u> A <u>was regarded as the ring of polynomial functions</u> $\mathfrak{F}_A(k)$, (which are continuous in the ZARISKI topology cf. next §14).

<u>If A is not reduced this functional interpretation fails</u>:

If we insist in calling a nilpotent $f \in A$ ($f \neq 0$) a "function" we have the paradoxical situation that <u>f is not zero but $\tilde{f}$ is identically zero</u>. Without the invention of sheaves it would be hard to overcome this objection.

SERRE proved in FAC that A is the ring of global sections of the structure sheaf $\mathcal{O}_X$ (X = Max A, A reduced), whose stalk at any $m \in$ Max A is the local ring A_m. When we remove the restriction on Nil A it is still possible to construct an $\mathcal{O}_X$ and it is still true that $A = \Gamma(X;\mathcal{O}_X)$ but it is <u>wrong</u> to call the elements of $\mathcal{O}_X$ germs of functions. In Ch. III GROTHENDIECK will tell us how for any commutative ring with unit A ($\in$ Ob$\mathfrak{C}$, cf. §2) by replacing Max A by the full spectrum Spec A it is possible to carry on all the constructions: the ZARISKI topology and the sheaf $\mathcal{O}_X$

It is not possible to get good results with the maximal spectrum.....

14. THE ZARISKI TOPOLOGY.

SUMMARY: The family $\mathcal{V}(k^n)$ of algebraic varieties in k^n (k algebraically closed field) contains the empty set $\emptyset$ and k^n and it is stable by finite unions and arbitrary intersections. Thus $\mathcal{V}(k^n)$ is the family of closed sets in a topology in k^n known as the ZARISKI topology.

Let us assume that the system of equations S consists of only one constant polynomial c. For $c = 0$ every point of k^n is a solution. $c \neq 0$ no point of k^n is a solution. Thus $\emptyset$, $k^n \in \mathcal{V}(k^n)$ (cf. Summar

Let $\{V_\nu\}_{\nu \in N}$ be an indexed set of algebraic varieties in k^n represented by systems of polynomial equations S_ν. ($\nu \in N$). Let $S = \bigcup_{\nu \in N} S_\nu$ be the union of all these equations. Then S represents the set-theoretic intersection of the V_ν

(14.1) $$V_S = \bigcap_{\nu \in N} V_\nu$$

The previous remarks hold actually for an arbitrary commutative ring with unit $\mathbb{1}$. Now if k is an integral domain and S_1, S_2 are two syst of polynomial equations we can see immediately that the system of equatio

(14.2) $$S = \{f_1 f_2 \mid f_1 \in S_1,\ f_2 \in S_2\}$$

represents the set-theoretic union:

(14.3) $$V_S = V_{S_1} \cup V_{S_2}$$

In fact $V_{S_1} \cup V_{S_2} \subset V_S$ is clear (for any k). On the other hand i

(14.4) $x \in V_S \quad (f_1 f_2)(x) = (f_1(x)) f_2(x) = 0 \Rightarrow$ either $f_1(x) = 0$ or $f_2(x) = 0$

(k is an integral domain).

Property (14.3) is extended to arbitrary <u>finite unions</u> by induction.

All these trivial properties were well-known in classical times,[1] but for many years this ZARISKI topology was not taken seriously because of the following highly undesirable properties:

1) It is very coarse!

2) The closed sets are very "thin".

2') Equivalently the open sets are "very large".

3) The closure of any non empty open set is the whole space.

4) Any two non empty open sets U, V intersect: $U \cap V \neq \emptyset$!

4) implies that the ZARISKI topology is non-Haussdorff for any $n > 0$. It is T_1 however, i.e. <u>every one-point set is closed</u> (it is the intersection of n linearly independent hyperplanes!).

The ZARISKI topology is coarser than the usual topology of $\mathbb{C}^n$; the pioneers used topological considerations based on this natural topology, for instance in conjunction with transcendental methods related to the theory of analytic functions.

SERRE'S FAC paper (1955) used ZARISKI topology instead of the natural topology of analytic manifolds, showing that many methods used before for analytic coherent sheaves can be extended to the purely algebraic case.

[1] Cf. for instance HODGE-PEDOE treatise: <u>Methods of algebraic geometry</u>, Cambridge Un. Press, 1947, Vol. II. The properties mentioned in the text are checked but no mention is made of the fact that this characterizes closed sets in a topology. Incidentally the same properties are true in projective space, with the only necessary modification that the polynomials of S need to be homogeneous.

We shall see in Ch. III how the ZARISKI topology in this original nai sense still preserves a meaning in much more general situation, where poin become prime ideals of some commutative ring with unit A. In fact (14.3 becomes, in a natural sense

$$(14.3)' \qquad "(f_1f_2)(p) = 0" \Rightarrow \text{either } "f_1(p) = 0" \text{ or } "f_2(p)"$$

where $f(p) = 0$ is used as a conventional notation for $f \in \mathfrak{p}$ or $f = 0$

CHAPTER II

LIMITS IN THE CATEGORY AFF_k

OF

AFFINE ALGEBRAIC SPACES

Ch. II can be omitted in the first reading. GROTHENDIECK needed thi material for his course on algebraic groups. The construction of the af scheme $(\mathrm{Spec}\, A, \tilde{A})$ attached to any commutative ring with unit A does r require any material in Ch. II. We advise the beginner to read Ch. III first.

...So we have set up the basic language concerning affine algebraic spaces cf. Ch. I, § and we could proceed now in two directions: either restating the notion in a different way that will allow us to glue together affine pieces and define more general algebraic spaces whi don't need to be affine, i.e. which don't need to be representable by k-algebras,(1) or we coul alternatively first work a little bit more in this category Aff_k of restricted objects defining such operations as products, fiber products, kernels, cf. §1, 6 . Since we are goi to introduce affine group objects,(2) i.e. group objects in Aff_k we choose this second altern tive before globalizing these notions defined in Vol. II.

SUMMARY. Chapter II is mainly devoted to the study of <u>limits</u> (cf. §2) in the category Aff_k of affine algebraic spaces over k $(k \in \mathrm{Ob}\,G)$, in particular, for $k = \mathbb{Z}$, in the "<u>absolute case</u>": $\mathrm{Aff} = \mathrm{Aff}_{\mathbb{Z}}$.

An object of Aff_k was already defined in Ch. I as the functor of solutions, when we get

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(1) ...which leads us to define the spectrum $X = \mathrm{Spec}\, A$ and the structure sheaf $\mathcal{O}_X$ and then we glue together several affine pieces Vol. II .

(2) This will be handy for defining group objects in the category Aff_k needed in his course on affine algebraic groups taught simultaneously in SUNY at Buffalo, Summer 1973, cf. AAG-Buffalo course.

rid of the affine embeddings by replacing S by $\mathcal{J}$ and then by A ($\in \mathrm{Ob}\,\mathfrak{A}_k$), where $A = P_I/\mathcal{J}$, and we were led to identify such a functor with the representable functor $\mathfrak{X} = \mathfrak{X}_A = V_A : k' \mapsto \mathfrak{X}_A(k')$ ($\forall k' \in \mathrm{Ob}\,\mathfrak{A}_k$) where A is "the k-algebra representing $\mathfrak{X}$".

The audience requested a purely categorical review of the fundamental concepts of inverse and direct limits, as well as the reduction to some simple particular cases (treated first in §1,2,3). Since the graduate teaching in commutative algebra and topology (both point-set and algebraic) became increasingly categorical there are many chances that a knowledgeable reader might skip the first purely categorical part which takes much more space than the actual applications to the two dual categories Aff_k and $\mathfrak{A}_k$. However it would be rewarding to get acquainted with these notions through GROTHENDIECK's very concrete introduction (based on TOHOKU). I found it much easier to read MACLANE's graduate text on categories after reading these notes.

PART I

CATEGORICAL PREPARATION

REFERENCES.
For further details cf. EGA, Springer, Ch. 0, §1, page 19, ; cf. also TOHÔKU, 1.7d), SGA, 3, I. Quicker introductions without proofs can be found in DIEUDONNÉ, Advances, II. A more technical exposition of this material is given in MACLANE's category text.

1. PRODUCTS, KERNELS, FIBER PRODUCTS.

We shall review these categorical notions, first in a pure set theoretic way, then in various simple topological or algebraic geometric (but classical) contexts. Finally GROTHENDIECK will tell us how these notions in their original set-theoretic form can be characterized and redefined just in terms of arrows (maps, morphisms), in a way that the same definitions will make sense in an arbitrary category C. These product constructions will appear as particular cases of inverse limits (cf. §2) and they will help us to understand the general definitions, that can be stated in terms of the notion of representable functor (cf. Ch. I, §1) by a reduction to the set-theoretic case.[1] It is important to keep in mind, that this reduction is done via sets of morphisms in C. Precisely, we are going to use the inclusion:

$$(1.1)\qquad i : C \hookrightarrow \hat{C} = \underline{\mathrm{Hom}}(C^*, \mathrm{Sets})$$

by replacing every object $X \in \mathrm{Ob}\,C$ with the representable functor $i(X) = h_X : C \to \mathrm{Sets}$.

Thus if we know what the set-theoretic product $A \times B$ or $\ker(u,v)$ means for a double set-theoretic arrow $A \underset{v}{\overset{u}{\rightrightarrows}} B$, then $Z = X \times Y$ or $Z = \ker(u,v)$ for $X \underset{v}{\overset{u}{\rightrightarrows}} Y$ ($X,Y,Z \in \mathrm{Ob}\,C$; $u,v \in \mathrm{Arr}\,C$) iff for every $T \in \mathrm{Ob}\,C$ we have:

$$(1.2)\qquad \mathrm{Hom}_C(T,Z) = \mathrm{Hom}_C(T,X) \times \mathrm{Hom}_C(T,Y)$$

or

$$(1.3)\qquad \mathrm{Hom}_C(T,Z) = \ker(u_*,v_*)$$

where u_*, v_* are the induced maps $\mathrm{Hom}_C(T,X) \underset{v_*}{\overset{u_*}{\rightrightarrows}} \mathrm{Hom}_C(T,Y)$. Of course (1.2), (1.3) make sense because the $\mathrm{Hom}_C(\ ,\ )$ are sets.

The important construction of fiber products can be reduced to the ordinary product and to $\ker(\ ,\ )$.

[1] Very often, but not always, an object of C is a set X with some kind of additional structure (for instance a group, ring, topological structure), thus there is an underlying set. We do not use the points of these underlying sets...(which do not appear in the categorical axioms...): the definitions need to be purely categorical. As a counterexample remember that any preordered set becomes a category, (where the objects are not sets with an additional structure).

PRODUCTS:

Let us start by recalling the definition of the usual Cartesian pro $X \times Y$ of two sets X, Y as set of pairs (x,y):

(1.4)
$$X \times Y = \{(x,y) \mid x \in X,\ y \in Y\}$$

There are two maps $p,q\colon X \times Y \xrightarrow{p} X$ and $X \times Y \xrightarrow{q} Y$, usually called projections on the first (second) factor defined by

(1.5)
$$p(x,y) = x \qquad q(x,y) = y \qquad \forall (x,y) \in X \times Y$$

KERNELS:

Let u,v be two maps $X \overset{u}{\underset{v}{\rightrightarrows}} Y$. The kernel of the pair (u,v) is defined by

(1.6)
$$\ker(u,v) = \{x \in X \mid u(x) = v(y)\}$$

The set of fixed points of a map $f\colon X \to X$ is obviously given by $\ker(f, 1_X)$.

REMARK. The usual ker, so frequent in homological algebra, is actually a particular case w the second map v is equal to "zero", in the following sense. Let us assume that we repl the category of sets by the category Set_0 of pointed sets, i.e. an object is a set $(\neq \emptyset$ with a distinguished base element 0 and the morphisms are base preserving set maps $X \to$ i.e. mapping 0_X in 0_Y. The map induced in the underlying sets for abelian groups, modules, rings are distinguished in the sense that the zero element is distinguished. In Set_0 the zero map $X \to Y$ is always defined and $\ker f = \ker(f,0)$ in the sense of the previous definition.

FIBER PRODUCTS

An important related notion is the fiber product $A \times_S B$ of two set theoretic maps: If $A \xrightarrow{f} C$, $B \xrightarrow{g} C$ are two maps on the category of sets the fibre product $f \times_C g$ denoted usually by $A \times_C B$ for short (although f,g play an essential role) is the subset of the usual product $A \times B$

characterized by:

(1.7) $$A \times_C B = \{(a,b) \in A \times B \mid f(a) = g(b)\}$$

An important particular case of the fiber product is <u>the intersection</u> $A \cap B$ of two subsets of C, when f, g become the canonical injections $A \hookrightarrow C$, $B \hookrightarrow C$.

Let $\Delta = \{(x,y) \in C \times C \mid x = y\}$ be the diagonal of $C \times C$. We obviously have

(1.8) $$\Delta = \ker(p_1, p_2)$$

where (p_1, p_2) denote the two projections $C \times C \overset{p_1}{\underset{p_2}{\rightrightarrows}} C$

The previous diagonal case can be interpreted as a particular case of the intersection when $A = B = C$, $f = g = 1_C$.

There is a natural (bijective) diagonal map $\delta: C \to \Delta$ $(x \mapsto (x,x))$, $\forall x \in C)$.

Then the intersection $A \cap B$ is expressed as follows:

(1.9) $$\delta(A \cap B) = A \times B \cap \Delta^{(1)}$$

The Cartesian product $X \times Y$ of two sets can be trivially identified with $X \times_{\{e\}} Y$ where $\{e\}$ is a one-point set.[2]

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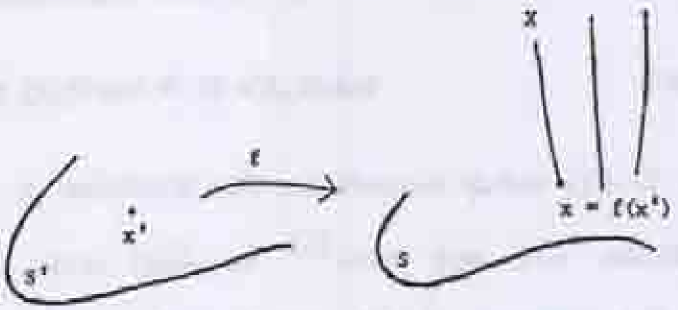
(1) In Classical algebraic Geometry, this construction has advantages when Δ becomes a <u>linear subspace of</u> $C \times C$ (for instance if C is an affine space). This <u>reduction to the diagonal or diagonal trick</u> has been widely used in the intersection theory of classical algebraic varieties.

(2) This will have a categorical analog for more general categories, depending on the fact that $\{e\}$ is a <u>final object</u> in the category of Sets. (Cf. I §1).

Of course, these pure set theoretical notions become more express when the sets have some additional structure. For instance, if X,Y a topological spaces, $X \times Y$ is usually understood to be the Cartesian product of the underlying sets endowed with the product topology. If X,Y are vector spaces over a field k of finite dimensions m,n , X can be naturally identified with $X \oplus Y$. In classical algebraic Geom if X,Y are irreducible projective algebraic varieties over $\mathbb{C}$ so i $X \times Y$ and the irreducible subvarieties of $X \times Y$ are by definition th irreducible correspondences between X and Y .

There is another interpretation of the fiber product $X \times_S Y$ as base change (widely used by GROTHENDIECK) which we need to discuss. Le $E(X \xrightarrow{p} S)$ be a fiber space of some sort (X is the total space, S t base space, p the projection).



Let $S' \xrightarrow{f} S$ be a continuous map (base change). We want to constr the inverse image $f^{-1}(E)$ (or pull-back) as follows: For every x' we assign as fiber over x' the fiber of E over $f(x') = x$ (i.e. " back the fibers"). In order to formalize and to introduce the topolo structure on $E' = f^{-1}(E)$, we construct first the topological product $X \times S'$ and then we restrict it to $X \times_S S'$; conversely any $A \times_C B$ can interpreted as a pull back of $B \to C$ by the base change $A \to C$. Now let us see how all these constructions might be introduced in an arbitrary category $\mathbb{C}$...Let us consider two objects X and Y of $\mathrm{Ob}\,\mathbb{C}$ and let us see what we can say about the product $X \times Y$ (that might not exist...)

If $X \times Y$ exists, it is a new object ($\in \mathrm{Ob}\,\mathbb{C}$). In order to define it axiomatically up to isomorphism, we shall examine first products of sets: If X, Y are objects of the category Sets, <u>the main categorical property of $X \times Y$ is</u> the existence of the two projections (1.5) in such a way that, for any other set Z giving a map $Z \xrightarrow{u} X \times Y$ is essentially the same thing as giving two maps $Z \to X$, $Z \to Y$ such that the diagram

(1.10)
$$\begin{tikzcd} & X \times Y \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \\ & Z \arrow[uu, "u"] \arrow[ul] \arrow[ur] & \end{tikzcd}$$

commutes. In other words we have a bijection $u \mapsto (p \circ u, q \circ u)$

(1.11)
$$\mathrm{Hom}(Z, X \times Y) \xrightarrow{\sim} \mathrm{Hom}(Z,X) \times \mathrm{Hom}(Z,Y)$$

(cf. (1.2)) which associates the morphism u with the ordered pair of morphisms $p \circ u$ and $q \circ u$.[1] We shall write $u = (p \circ u) \times (q \circ u) = (p \circ u, q \circ u)$.

<u>The property</u> (1.11) <u>will be taken as definition of the product $X \times Y$ for any category</u> $\mathbb{C}$: Precisely:

The object $X \times Y \in \mathrm{Ob}\,\mathbb{C}$ together with two arrows $p, q : X \times Y \xrightarrow{p} X$, $X \times Y \xrightarrow{q} Y$ is called <u>a product of the objects X, Y</u> (both $\in \mathrm{Ob}\,\mathbb{C}$) when p, q are universal with respect to all ways of mapping any $Z \in \mathrm{Ob}\,\mathbb{C}$ into $X \times Y$,

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[1] By the way, if Z is reduced to a point $[e]$ this just means that $X \times Y$ consists of all the pairs (x,y) $(x \in X, y \in Y)$, because to define a map $[e] \to Z$ is to choose a point of Z; and in particular, $[e] \to X \times Y$ amounts to choosing a pair (x,y),...

i.e. iff for every $Z \to X \times Y$ the map defined by $u \to (p \circ u, q \circ u)$ between $\mathrm{Hom}(Z, X \times Y)$ and $\mathrm{Hom}(Z,X) \times \mathrm{Hom}(Z,Y)$ is bijective.

REMARK. Since $\mathrm{Hom}(Z,X \times Y)$, $\mathrm{Hom}(Z,X)$, $\mathrm{Hom}(Z,Y)$ are sets, essentially, the definition of a product as a solution of a universal problem is redu to the particular case of the category of sets. This solution is define up to isomorphism, i.e. if $(X \times Y)'$ is a second product, one can find a arrow $u: (X \times Y)' \to (X,Y)$ commuting with the projections; reversing the role of $X \times Y$ and $(X \times Y)'$ we see that this is indeed an isomorphism.

One interesting problem on any category which we may consider, is t find out whether or not there exists such a categorical product $X \times Y$ any pair of objects X, Y of $\mathrm{Ob}\,C$. This product might exist or not b if it exists it is unique up to isomorphism.[1]

Namely, we considered the case of two objects before; but we can a take a family of objects $X_i \in \mathrm{Ob}C$ depending on one index $i (\in I)$ and to define the product $\prod_{i \in I} X_i$ of the X_i. This should be endowed with a family of morphisms p_i of the product into each of the X_i:

$$(1.5)' \qquad \prod_{i \in I} X_i \xrightarrow{p_i} X_i$$

and this would have the universal property that, for every $Z [\in \mathrm{Ob}C]$, th map which goes from $\mathrm{Hom}(Z, \prod_{i \in I} X_i)$ into the product $\prod_{i \in I} \mathrm{Hom}(Z, X_i)$, wh associates with a morphism $u: Z \to \prod_{i \in I} X_i$, the system of all composita should be bijective. In symbols, we have:

$$(1.11)' \qquad \mathrm{Hom}(Z, \prod_{i \in I} X_i) \xrightarrow{\sim} \prod_{i \in I} \mathrm{Hom}(Z, X_i)$$

[1] This product formation is a particular case of the construction of the so-called inverse limits in a category. (Cf. §2 for the general definition).

So this is the case of the product of an arbitrary family! One can wonder whether or not this product exists in C , but if it does exist, it i defined up to a unique isomorphism.

By the way, when the family is empty $\lceil I = \emptyset \rfloor$ we can wonder whether such a "product of an empty family exists" $\lceil$in order that (1.11)' still hol conversely: it is easy to show that the product becomes a <u>final object</u>, (cf. Ch. I, §1).

Thus, among the objects obtained by product formations, we also have the construction of final objects! Of course, when I contains just one element, the product over I always exists in any C , but this is not so for $\#|I| > 1$. course, if the product exists for any pair of objects, then we can always prove the existence of products of n objects (for any finite n) by pure associativity considerations.

Now let us see how we can extend to any C the notion of <u>kernel of a double arrow</u>: Let us assume that we have two arrows $X \underset{q}{\overset{p}{\rightrightarrows}} Y$ in a category C with the same source X and the same target Y . We want to define the kernel K of these two arrows (where $K \in \mathrm{Ob}\,C$) , by a reduction to the set theoretic case <u>involving arrows only</u>. A kernel would be an object $K \lceil \in \mathrm{Ob}\,C \rfloor$ together with a morphism $i: K \to X$ (as an extra structure). Composing i with the two given arrows, we have two morphisms from K into Y such that $p \circ i = q \circ i$ and <u>we want this arrow i to be universal</u>, for all arrows with target in X , relative to the property that the composites with p and q are equal. I will make this more explicit: for any object $Z (\in \mathrm{Ob}\,C)$, we look at the set Hom(Z,K) of all morphisms from Z into K ;

by composing with i this is mapped into the set of morphisms from Z to X: $f \mapsto i \circ f$; and composing again with p and with q, I get two maps $p \circ i \circ f$, $q \circ i \circ f$ from Z into Y. Then <u>(K,i) is a kernel for the pair</u> (p,q): $X \underset{q}{\overset{p}{\rightrightarrows}} Y$, iff for every choice of $Z (\in \mathrm{Ob}\,\mathcal{C})$ the sequence

$$\mathrm{Hom}(Z,K) \to \mathrm{Hom}(Z,X) \rightrightarrows \mathrm{Hom}(Z,Y)$$

is exact, i.e. the first arrow $\to$ is injective and its image consists of the morphisms from Z to X whose composition with both p and q are the same.

<u>IMPORTANT REMARK</u>. In the previous constructions of $X \times Y$ and of $\ker(X \rightrightarrows Y)$ by means of solutions of universal problems, <u>we reduced the</u> case of an arbitrary category $\mathcal{C}$ to the category of sets by replacing the object $X \in \mathrm{Ob}\,\mathcal{C}$ by the sets $\mathrm{Hom}_{\mathcal{C}}(Y,X)$ (for any $Y \in \mathrm{Ob}\,\mathcal{C}$). In other words, we were able to reduce the problem for an arbitrary $\mathcal{C}$, to the category of sets just by the substitutions $X \mapsto \mathrm{Hom}(\ ,X)$!!

Now, I will give two specific examples. Let us take the category of groups for instance. Here, there are products (the usual products of groups) and there are kernels also, because if $X \underset{q}{\overset{p}{\rightrightarrows}} Y$ represent two group homomorphisms the set $\{x \in X \mid px = qx\}$ is a subgroup $K = \ker(\underset{q}{\overset{p}{\rightrightarrows}})$, which turns out to be a kernel in the categorical sense!

Now let us give a third construction which is important.

FIBER PRODUCTS IN $\mathcal{C}$: Let us assume that we have two objects X, Y lying over a third one S i.e. we have two structural morphisms f,g (cf. Ch. I, §1):

$$(1.12)\qquad \begin{tikzcd} X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

We want to define a third object of C lying also over S, which we call the fibre product of X and Y with respect to S[(1)]; and we denote by $X \times_S Y$, which for the category of sets, $X \times_S Y$ should be the subset of the ordinary set-theoretic product $X \times Y$ consisting of all pairs which have the same projections by f and g (Cf. (1.7)). $X \times_S Y$ is nothing else as the usual product in the category C/S. In other words the fiber product, denoted usually by $X \times_S Y$ for short is actually a morphism $X \times_S Y \to S$ in C making commutative the left diagram below

$$(1.13)\qquad \begin{tikzcd} X \arrow[dr] & X \times_S Y \arrow[l] \arrow[r] \arrow[d] & Y \arrow[dl] \\ & S & \end{tikzcd} \qquad\qquad \begin{tikzcd} X \arrow[dr] & Z \arrow[l] \arrow[r] \arrow[d] & Y \arrow[dl] \\ & S & \end{tikzcd}$$

in such a way that for every $Z \to S$ in C making commutative the right diagram above it is possible to fill in the dotted arrow below in a unique way to construct a commutative diagram

$$(1.14)\qquad \begin{tikzcd} & Z \arrow[dl] \arrow[dr] \arrow[d, dashed] & \\ X \arrow[dr] & X \times_S Y \arrow[l] \arrow[r] \arrow[d] & Y \arrow[dl] \\ & S & \end{tikzcd}$$

REMARKS:

1) The fiber product $X \times_S Y$ can be defined in terms of products and kernels. If we have $X \xrightarrow{f} S$, $Y \xrightarrow{g} S$ we construct $X \times Y$ in C (if it

(1) This is an abuse of language (and $X \times_S Y$ an abuse of notation). f and g are essential.

exists) and then we construct (if possible): $\ker(X \times Y) \underset{(g \times g)\circ q}{\overset{(f \times f)\circ p}{\rightrightarrows}} S \times S$.

Thus <u>the existence of products and kernels imply the existence of fiber-products</u>

[1]Conversely if arbitrary fiber products exist in $\mathcal{C}$, there exist final object $\mathcal{C}$ ($\Leftrightarrow$ product of an empty family of objects of $\mathcal{C}$, cf. pag 56) and then $X \times Y = X \times_e Y$. Similarly we can see that there also exists arbitrary products. If arbitrary finite products and finite products exist in $\mathcal{C}$ we can define the diagonal $\Delta_X = X \times_X X$, the diagonal morphi $\delta_X : X \to \Delta_X$. The silly commutative diagram below

(1.15)
$$\begin{tikzcd}
& X \times_X X \arrow[dl] \arrow[d] \arrow[dr] & \\
X \arrow[r, "1_X"'] & X & X \arrow[l, "1_X"]
\end{tikzcd}$$

shows that δ_X is bijective and we have two canonical maps inverse to each other $X \rightleftarrows \Delta_X$. Now the pair of arrows $X \underset{v}{\overset{u}{\rightrightarrows}} Y$ can be transformed i the unique arrow $X \xrightarrow{(u,v)} Y \times Y$ and we can close a commutative diagram bel by completing the dotted arrows:

(1.16)
$$\begin{tikzcd}
K \arrow[r] \arrow[d, dotted] & Y \arrow[d, "{(1_Y,1_Y)}"] \\
X \arrow[r, "{(u,v)}"'] & Y \times Y
\end{tikzcd}$$

Then $K = \ker(u,v) = X \times_{Y \times Y} Y$ with respect to the two morphisms (u,v) and $(1_Y, 1_Y)$. This is an abstract version of the diagonal trick (inter section of the graph of (u,v) and the diagonal.

Now let us see how fiber products can be interpreted as <u>base change</u>

in an arbitrary category, following GROTHENDIECK's own words:⌋

Let us interpret the fiber products alternatively as being an operation of base change. Let us consider two objects X, S' (which will not play the same role) lying over S ⌈i.e. we consider two morphisms as shown in the diagram below:

$$\begin{tikzcd} & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

, where X can be viewed as a kind of "fiber space over S" and S' is viewed as a "new basis" (cf. the set-theoretic pull-back interpretation). Let $X' = X \times_S S'$ be the fiber product of X and S' over S. There are natural arrows of X' over X and over S' ⌈making commutative the diagram

(1.17)
$$\begin{tikzcd} & X' = X\times_S S' \arrow[r] \arrow[d] & X \arrow[d] \\ Z' \arrow[r, dashed] & S' \arrow[r] & S \end{tikzcd}$$

The arrow $X' \to S'$ defines X' as an "<u>object lying over S'</u>".(1) I want to characterize X' over S' in this yoga ⌈for $S' \to S$ fixed and X variable⌋. The solution will be a kind of tautological reformulation ⌈of the definition of $X \times_S S'$⌋ when the "optic" has been changed a little bit.

I look at the two categories C/S and C/S. ⌈cf. Ch. I, §1⌋ ; there are two natural functors φ, ψ going in opposite directions:

(1.18)
$$C/S' \underset{\psi}{\overset{\varphi}{\rightleftarrows}} C/S \qquad (X \overset{\psi}{\mapsto} X',\ Z' \overset{\varphi}{\mapsto} Z)$$

where φ is the composition functor $C/S' \overset{\varphi}{\to} C/S$, i.e. whenever I have an object $Z' \to S'$ over S' I map it into the composite $\varphi(Z') \to S$ as

(1) i.e. something corresponding to the intuition of a fiber space over X'.

indicated below (i.e. $Z' \to S'$ goes to $(Z' \overset{\varphi}{\to} S) = \varphi(Z')$ [1]. Now the

$$\begin{tikzcd} Z' \arrow[d] \arrow[dr, "\varphi"] & \\ S' \arrow[r] & S \end{tikzcd}$$

construction $X \times_S S'$ can be viewed as <u>a right adjoint</u> of
Precisely, we have the adjunction relation

$$\mathrm{Hom}_{C/S}(\varphi(Z'),X) \cong \mathrm{Hom}_{C/S'}(Z',\psi(X)) \;{}^{(2)}$$

we leave the trivial verification to the reader; the important point th GROTHENDIECK wants to make clear is the following:

Let $S' \to S$ be a fixed morphism ("<u>base change</u>" in an arbitrary category C). Then the composition functor $\varphi\colon C/S' \to C/S$ $(\Leftrightarrow Z' \to \varphi($ has a right adjoint [3] $\psi\colon X \to \psi(X) = X' = X \times_S S'$ which is the <u>fiber prod</u> or pull-back .

In other words <u>in any C any fiber products can always be interpr</u> <u>as a base change.</u>[4]

2. REDUCTION OF ANY INVERSE LIMIT TO THE PREVIOUS PARTICULAR CASES.

SUMMARY. We are going to define the <u>inverse limit</u>[5] of an arbitrary diagram in the category C. Such diagrams appear as concrete realizations in C of an abstract diagram D of vertices and arrows. The previous constructions of $X \times Y$, $X \times_S Y$, $\ker(u,v)$ will appear as particular cases. The inverse limit of any diagram D in C, $F = \varprojlim D$ always exists in

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(1) The consideration of the diagram $S' \to S \leftarrow X$ is asymmetric...

(2) Thus φ is a kind of forgetful functor (forgetting S' for the benefit of S).

(3) [The reader should recall that the notations X, Z' are abusive. Really the objects of C/S, C/S' are structure arrows $X \to S$, $Z' \to \underline{S'}$] .

(4) [I recall the (officially well-known) notion of adjonction. Let F,G be two functors running in opposite directions between two fixed categories $\mathcal{D},\mathcal{E}$:

$$\mathcal{D} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{E}$$

Let us assume that for every pair of objects $D \in \mathrm{Ob}\,\mathcal{D}$, $E \in \mathrm{Ob}\,\mathcal{E}$ there is a functorial bijection (called <u>adjonction</u>):

$$\Phi_{D,E}\colon \mathrm{Hom}_{\mathcal{D}}(E,G(E)) \cong \mathrm{Hom}_{\mathcal{E}}(F(D),E)$$

. Then we say that F (G) is a <u>right</u> (<u>left</u>) adjoint to G (F) . Usually $\Phi_{D,E}$ is clear enough (as in the text and it is not made explicit. Cf. MACLANE's <u>Category text</u> Ch. IV <u>Adjoints</u>, page 77.]

(5) Pull-backs in American categorical parlance.

the enlarged category $\hat{C}$, but it does not need to be an object of C. It is just a contravariant functor from C to Sets. When F belongs to the essential image of C in $\hat{C}$ by the inclusion $i: C \hookrightarrow \hat{C}$, in other words when F is representable (for instance $F = h_X$, $(X \in \mathrm{Ob}\, C)$) then we write $X = \varprojlim D$, but then X becomes an object of C determined up to isomorphism.

We shall show how the general inverse limits in C can be reduced to the cases of §1.

ABSTRACT DIAGRAMS.

Let us start with a diagram D, which is just a set of vertices D_0 and a certain set of arrows D_1 "joining certain pairs of D_0" (something very abstract!) and two maps from D_1 into D_0:

$$D_1 \rightrightarrows D_0 \tag{2.1}$$

associating to every arrow two vertices (its origin and its extremity) respectively, also frequently called the source and the target of the arrow. This is what we call a model for the diagram.[1]

Now, let us consider a category C. Then a diagram of type D in C is just a pair of maps, the first one associating with every vertex of D an object of C: $D_0 \to \mathrm{Ob}\, C$ and the second one associates with every abstract arrow of D a morphism of $\mathrm{Arr}\, C = C_1$ [2] preserving sources and

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(1) Cf. for further reading MACLANE Category text, III Universals and Limits, page 55. GROTHENDIECK used in his course the terminology inverse and direct limits. The following dictionary can be useful:

Inverse limit = projective limit = limit = left roots = $\varprojlim$

Direct limit = inductive limit = colimit = right roots = $\varinjlim$

GROTHENDIECK used the French terminology (projective, inductive limits) in his original papers. Cf. for instance SGA_4-I, Exposé I (Préfaisceaux).

(2) Such abstract diagrams are MACLANE's Graphs (loc. cit, II, §7, p. 48). "Every category determines a graph (= diagram) forgetting which arrows are composites and which are identities". Conversely there are ways to "generate" categories from diagrams by freely composing the arrows in some way. Cf. loc. cit. for details.

targets, i.e. in such a way that the following square commutes

(2.2)
$$\begin{tikzcd} C_1 = \mathrm{Arr}C \arrow[d, shift left] \arrow[d, shift right] & D_1 \arrow[l] \arrow[d, shift left] \arrow[d, shift right] \\ C_0 = \mathrm{Ob}\,C & D_0 \arrow[l] \end{tikzcd}$$

EXAMPLES. 1) Let us first consider the case of two points: • • (no arrows). A diagram of this type in C consists in giving just two objects X and Y, in the category C.

2) Consider a diagram with two points and two arrows in the same direction: $\bullet \rightrightarrows \bullet$. It corresponds to a pair X,Y together with two morphisms from X into Y : $X \rightrightarrows Y$

3) In order to define fibre products $X \times_S Y$ we shall consider three vertices 1,2,3 and two arrows 1-3 and 2-3 with the same target.

(2.3)
$$\begin{tikzcd} 1 \arrow[dr] & & 2 \arrow[dl] \\ & 3 & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

corresponding to three objects X,Y,S of C.

⌈ I summarized here the previous examples of §1:

	PRODUCT	KERNEL	FIBRE PRODUCT
Diagram type	• •	• •	• • •
DIAGRAM in C	X Y	$X \rightrightarrows Y$	$X \to S \leftarrow Y$
$\varprojlim$	$X \times Y$	$k = \ker(X \rightrightarrows Y)$	$X \times_S Y$

⌋

Now we want to define the inverse limit of any type of diagrams in the category $\mathbb{C}$. $\Leftrightarrow$ abstract diagrams . I will now use small letters d, d_0, d_1 to define types of diagrams and capitals D, D_0, D_1 to describe effective diagrams.

Let D be a diagram of type d in the category $\mathbb{C}$. D is defined by associating with any i $(\in d_0)$ an element $D(i) \in \mathrm{Ob}\,\mathbb{C}$ and with every arrow φ $(\in d_1)$ a morphism $D(\varphi) \in \mathrm{Arr}\,\mathbb{C}$, satisfying the commutativity property (2.2).

We call inverse limit $(\varprojlim D)$ of such a diagram D an object $L \in \mathrm{Ob}\,\mathbb{C}$ together with a family $u(i)$.

(2.4) $$L \xrightarrow{u(i)} D(i) \qquad i \in d_0$$

of morphisms of L into the $D(i)$ satisfying the following axiom:

For every object $Z \in \mathrm{Ob}\,\mathbb{C}$ of the category and every morphism $Z \to L$ we define a map

(2.5) $$\mathrm{Hom}(Z,L) \to \prod_{d_0} \mathrm{Hom}(D,D(i))$$

by composition of arrows

(2.6) $$\begin{tikzcd} L \arrow[r, "u(i)"] & D(i) \\ Z \arrow[u] \arrow[ur] & \end{tikzcd}$$

in such a way that (2.5) is injective and the image should be exactly the set of all morphisms from Z into the various $D(i)$ which is compatible with all the arrows of the diagram, i.e. if $D(i) \to D(j)$ belongs to D then the triangle

(2.7)

$$\begin{tikzcd} D(i) \arrow[r, "D(\varphi)"] & D(j) \\ Z \arrow[u] \arrow[ur] & \end{tikzcd}$$

is commutative.

A set of morphisms $Z \to D(i)$ satisfying the previous commutativity condition for all the $D(\varphi) \in D_1$ is called a coherent set of morphisms.

Then, we can express the previous property stating that (2.5) defin a bijective map between $\mathrm{Hom}(Z,L)$ and the set of coherent sets of morphisms in D, i.e.

(2.8)
$$\mathrm{Hom}(Z,L) \rightrightarrows \prod_{\substack{i \in d_0 \\ \text{coherent}}} \mathrm{Hom}(Z,D(i))$$

This can be said also in the following way, by reducing the general case to the category of sets as follows:

Let us assume $C = \text{Sets}$; then all the $D(i)$ are sets and the morph $D(i) \overset{\varphi}{\to} D(j)$ are maps between the corresponding sets. Then we can look at the subset of the product of the $\mathrm{Hom}(Z,D(i))$ which consists of sets of morphisms compatible with the arrows of D and commutes for every $D(\varphi)$ of the diagram). Then the universal property defining $\varprojlim$ of any diagram D of type d in a category C can be stated as follows:

An object $L \in \mathrm{Ob}C$ is the $\varprojlim D(i)$ of the diagram D of type d iff there is a bijective correspondence with the limit of the $\mathrm{Hom}(Z,D(i))$ (of type D) (in the category of sets):

$$\text{(2.9)} \qquad \mathrm{Hom}(Z,L) \xrightarrow{\sim} \varprojlim \mathrm{Hom}(Z,D(i))$$

<u>for every choice of the object Z of $\mathrm{Ob}\,\mathbb{C}$</u>.

Now, we are going to prove the reduction theorems:[1]

<u>Let $\mathbb{C}$ be an arbitrary category. The following properties are equivalent</u>:

1) <u>Arbitrary inverse limits exist in $\mathbb{C}$</u>

2) <u>$\mathbb{C}$ contains arbitrary products and arbitrary kernels for pairs of arrows.</u>

3) <u>$\mathbb{C}$ contains arbitrary products and arbitrary fibre products.</u>

PROOF: It is sufficient to remark the existence of a functorial isomorphism

$$\text{(2.10)} \qquad \varprojlim D \simeq \ker\Big(\prod_{i\in d_0} D(i) \overset{u}{\underset{v}{\rightrightarrows}} \prod_{j\in d_1} D(\text{target of } j)\Big)$$

for a diagram D, where the two products are taken over d_0 (set of vertices) and d_1 (set of arrows). The two arrows of (2.10) are defined, by replacing with unique arrows u,v, the two sets of arrows

$$\prod_{i\in d_0} D(i) \xrightarrow{pr_{\text{target } j}} D(\text{target } j)$$

and the composite of

$$\prod_{i\in d_0} D(i) \xrightarrow{pr_{\text{source } j}} D(\text{source } j) \xrightarrow{D(j)} D(\text{target } j)$$

REMARK. Since the conditions 2), 3) are satisfied in the category of Sets

[1] We borrow the explicit description of this reduction from SGA-4-I, <u>Exposé I</u>, page 11. It should be taken with the usual grain of salt because we do not care here about the foundations of category theory, universes, etc. Besides the abstract diagrams d can be replaced by a suitable "small" category $\mathcal{S}$ and the concrete diagrams by functors $\mathcal{S} \to \mathbb{C}$.

we know the existence of arbitrary inverse limits in the category of set

Now after this preparation, we recognize easily that, for any given category C the $\varprojlim D$ always exists in the category $\hat{C}$, since the construction is reduced elementwise to the category of sets. We say tha $\varprojlim D$ exists in C iff the limit of D in $\hat{C}$ is representable, i.e. iff $\varprojlim D$ belongs to the essential image of $i: C \hookrightarrow \hat{C}$. Let $\varprojlim D \simeq h_L$ $(L \in \mathrm{Ob}\, C)$. Then the object L (defined up to isomorphism i C) is called an inverse limit of D in C and we shall abuse the notati by writing $L = \varprojlim D$.

3. DIRECT LIMITS. The direct limits in a category C (cf. footnote (1) II-2-2 page 62) correspond bijectively to the inverse limits in the oppos category C°. These duality considerations are particularly important si the main result of Vol. I is that the category Aff of affine algebraic spaces (that will be identified in Ch. III with the category of affine sc is isomorphic with the opposite category G° of the category G of commutative rings with unit.

Because of this fact, although we could honestly leave to the reader the main facts regarding direct limits, we shall elaborate a little bit or certain basic notions, by dualizing explicitly the constructions of (II.1) and (II.2), i.e. we shall start by dualizing products, kernels, fiber products as sums (= coproducts), cokernels and amalgamated sums.

SUM = COPRODUCT. The basic set theoretic remark is the discovery b the pioneers that the disjoint sum of two sets A, B (union of two disjoin copies): $A + B$ $(= A \amalg B)$ has the dual properties of $A \times B$, i.e. to the projections $A \times B \to A$, $A \times B \to B$ correspond the canonical injections:

$A \hookrightarrow A + B$, $B \hookrightarrow A + B$ and the identification (II, (1.2)) is replaced by

$$(3.1) = (1.2)' \qquad \mathrm{Hom}_{\mathbf{C}}(A + B, C) = \mathrm{Hom}_{\mathbf{C}}(A,C) \times \mathrm{Hom}_{\mathbf{C}}(B,C)$$

defined by filling in the dotted arrow below in order to make commutative the dia

$$(3.2) \qquad \begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ A + B \arrow[rr, dashed] & & C \\ & B \arrow[ul] \arrow[ur] & \end{tikzcd}$$

(dualize (II-1-10) or (II-1-11)). The property (3.1) is taken as the definition of $A + B$ ($= A \amalg B$) of the sum ($=$ coproduct) of two objects A,B in any category $\mathbf{C}$.

The extension to any family $\{X_i\}_{i \in I}$ of objects in $\mathbf{C}$ is done accordingly by dualizing (1.11)'

$$(3.3) \qquad \mathrm{Hom}_{\mathbf{C}}(\coprod_{i \in I} X_i, Z) \rightrightarrows \prod_{i \in I} \mathrm{Hom}_{\mathbf{C}}(X_i, Z)$$

As for the product one can wonder whether or not the sum exists in $\mathbf{C}$ but if it does exist, it is defined up to a unique isomorphism.

DUALIZATION OF THE KERNEL (COKERNEL = COEQUALIZER = coker $(X \overset{f}{\underset{g}{\leftleftarrows}} Y)$). The universal injective arrow $K \xrightarrow{i} X \overset{f}{\underset{g}{\rightrightarrows}} Y$ ($f \circ i = g \circ i$) is dualized in our case $X \overset{f}{\underset{g}{\leftleftarrows}} Y$ (sense of the arrows reversed) by a pair $(Q, p: Q \leftarrow X)$ such that $Q \overset{p}{\leftarrow} X \overset{f}{\underset{g}{\leftleftarrows}} Y$ commutes and for any commutative diagram $Z \overset{p'}{\leftarrow} X \overset{f}{\underset{g}{\leftleftarrows}} Y$ there should exist a unique morphism $Q \xrightarrow{\varphi} Z$ such that $p' = \varphi \circ p$. In other words Q looks like a universal quotient. The corresponding pairs of dual diagrams indicate clearly the names equalizers and coequalizers given by professional categorists to kernels and cokernels. Geometrically $\ker(X \overset{f}{\underset{g}{\rightrightarrows}} Y)$ arises by considering $\{x \in X | f(x) = g(x)\}$. We would like to change the role of X and Y .

This is not literally possible. What we obtain (instead of a subset K o X $(K \overset{i}{\hookrightarrow} X)$ is a quotient object Q $(Y \overset{p}{\rightarrow} Q)$ such that the compositions of the two arrows $X \overset{f}{\rightarrow} Y$, $X \overset{g}{\rightarrow} Y$ with p become identical: $X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{p} Q$ commutes

DUALIZATION OF FIBER PRODUCTS: AMALGAMATED SUMS. The dual of a fiber product $X \times_S Y$ (= a product of two objects in the category C/S) will be a sum in the category S/C. It is usually called an amalgamated sum in C. The best example is the usual union $A \cup B$ of two subsets $A \hookrightarrow$ $B \hookrightarrow S$ of a set S. We can obtain $A \cup B$ dividing out $A \amalg B$ by the equivalence relation defining the identification $a \sim b$ ($a \in A$, $b \in B$ iff $a = b$).

The extensions of these notions to arbitrary families $\{X_i\}_{i \in I}$ can b done directly, i.e. dualizing (1.5)', (1.11)' (cf. page 55)

$$X_i \rightarrow \coprod_{i \in I} X_i \qquad \mathrm{Hom}_C(\coprod_{i \in I} X_i, Z) \overset{\sim}{\rightarrow} \prod_{i \in I} \mathrm{Hom}_C(X_i, Z)$$

and the sum of an empty family of objects of C is an initial object (cf. I, §1).

A good exercise would be to see how the existence of arbitrary finite sums and cokernels imply the existence of amalgamated sums, etc., and finally the reduction of arbitrary direct limits to the previous particular cases.

The device of reducing the $\varprojlim$ in any category C to the category c sets is done by enlarging C to the category $\hat{C}$ of contravariant functor $C \rightarrow$ Sets, where every $X \in \mathrm{Ob}\,C$ was identified with the functor h_X represented by X. A similar approach in $\hat{C}^\circ$ enables the reduction of the study of direct limits to the case of sets.

II

LIMITS IN THE CATEGORY AFF_k

SUMMARY. The inverse limit constructions in Aff_k are deduced by duality from the direct limits in the opposite category $\mathbb{G}_k$. The main property is the existence of arbitrary sums (= coproducts) in $\mathbb{G}_k$ coming essentially from the tensor product $A \otimes_k B$ (= categorical sum of A and B, $A, B \in \mathrm{Ob}\,\mathbb{G}_k$). cf. §6. In $\mathbb{G}_k$, arbitrary cokernels of pairs of arrows exist. (cf. §3). As a consequence, in Aff_k arbitrary products and kernels $k(X \rightrightarrows Y)$ of arbitrary pairs of arrows exist and, because of the reduction theorem of §2, we can prove the existence of arbitrary inverse limits in the category Aff_k.

Regarding direct limits $\varinjlim X_i$ in $\mathbb{C} = \mathrm{Aff}_k$, we know that they correspond to inverse limits $\varprojlim A_i$ $\lceil X_i = \mathfrak{X}_{A_i}$, $\forall i \in \underline{I}\rfloor$ in the opposite category $\mathbb{G}_k$. $\lceil A_i \in \mathrm{Ob}\,\mathbb{G}_k$, $\forall i \in \underline{I}\rfloor$

These inverse limits always exist in $\mathbb{G}_k$, since the set-theoretic limit of the underlying sets of the A_i always exist and the algebraic structure carries over. The trouble arises when we want to interpret them in the opposite category $\mathbb{C}$, i.e. the dual direct limits in $\mathbb{C}$ are not too reasonable! For instance, passage to quotients in Aff_k gives pathological results because it would be not the same as passage to quotients in the more general category of schemes (cf. Vol. II).

There is just one operation, which is always reasonable: the finite sum operation $\coprod_{i \in I} X_i$ $\lceil I = \{1, 2, \ldots, n\}$ finite index set$\rfloor$ in the category $\mathbb{C}$ of affine algebraic spaces over k that we are going to describe in detail below.

4. $\lceil$CATEGORICAL GENERALITIES ON AFFINE ALGEBRAIC SPACES. The two categories Aff, (Aff_k) of affine algebraic spaces (a.a. spaces over k, $k \in \mathrm{Ob}\,\mathbb{G}$) have been defined already in Ch. I by a canonical identification with the categories $\mathbb{G}^*$ ($\mathbb{G}_k^*$) opposite to the categories $\mathbb{G}$, ($\mathbb{G}_k$) of commutative rings with unit (k-algebras, ($k \in \mathrm{Ob}\,\mathbb{G}$).) More precisely $\mathbb{G}^*$ and $\mathbb{G}_k^*$ are identified with the category of covariant representable functors (cf. Ch. I, §1):

$$\mathfrak{X}_A : \mathbb{G} \to \mathrm{Sets} \qquad (\text{or } \mathbb{G}_k \to \mathrm{Sets}) \tag{4.1}$$

for every $A \in \mathrm{Ob}\,\mathbb{G}$ (or $A \in \mathrm{Ob}\,\mathbb{G}_k$). Besides we have $\mathrm{Aff} = \mathrm{Aff}_{\mathbb{Z}}$.

From these definitions follow many simple consequences of a pure categorical nature regarded as "general abstract nonsense"; I summarized a few of them, just to show how easily we can deduce so many properties just by "reversing arrows". But I am well aware that some old-fashioned reader will ask, why this is called geometry! The most beautiful answer is the introduction of the affine schemes (cf. Ch. III), i.e. by giving and interpretation of $\mathfrak{X}_A$ by a geometric object, bijectively attached to $\mathfrak{X}_A$ the <u>affine scheme</u> Sch $A = (\mathrm{Spec}\ A, \tilde{A})$ (a certain topological space: the Spectrum of A (Spec A) with a structure sheaf of local rings $\tilde{A}$). But even without that, it is possible, just in the framework of representable functors, to answer to all criticisms by showing how all the specific properties of the categories $\mathfrak{C}$, $\mathfrak{C}_k$ <u>have geometrical interpretations lacking in other categories</u>.

1) GENERAL ABSTRACT NONSENSE: The map $A \mapsto \mathfrak{X}_A$ is a bijection $\mathrm{Ob}\,\mathfrak{C} \to \mathrm{Ob}\ \mathrm{Aff}$. If $A \xrightarrow{f} B$ is a morphism in $\mathfrak{C}$, $f : \mathfrak{X}_B \to \mathfrak{X}_A$ is a morphism in Aff. $A \xrightarrow{1_A} A$ goes into the identity of $\mathfrak{X}_A$. Compositions are "anti-preserved": $A \xrightarrow{f} B \xrightarrow{g} C \Rightarrow \mathfrak{X}_C \xrightarrow{g^*} \mathfrak{X}_B \xrightarrow{f^*} \mathfrak{X}_A$. In other words, there is a contravariant functor $\mathfrak{C} \to \mathrm{Aff}$ which defines actually an isomorphism between Aff and the opposite category $\mathfrak{C}^{\circ}$. (Similar situation $\mathfrak{C}_k \to \mathrm{Aff}$ for any $k \in \mathrm{Ob}\,\mathfrak{C}$).

The relationship between $\mathfrak{C}$ and $\mathfrak{C}_k$ is a particular case of what we stated in §1 for any category C. Precisely $\mathfrak{C}_k$ is the same as the category $k/\mathfrak{C}$ whose objects are the ring homomorphisms $k \to A$ with variable target A ($\in \mathrm{Ob}\,\mathfrak{C}$). <u>Thus the algebraic spaces over k can be</u>

identified with relative objects of Aff/Ξ_k . Precisely, dualizing $k \overset{u}{\to} A$ we have $\Xi_A \overset{u_*}{\to} \Xi_k$ (and conversely). In other words Ξ_k *is the final object of the category* Aff_k , since *it is the dual of the initial object of* $\mathbb{G}_k$. (Cf. Ch. I, §1).

2) GEOMETRY: We will appeal to geometric intuition (I still like that!) to determine what the points of this space Ξ_A are? Let us recall Ch. I, §3:

An R-valued ($R \in \mathrm{Ob}\,\mathbb{G}$) *point* x *of* Ξ_A *is a ring homomorphism* $x: A \to R$ *and conversely*. (Cf. §1) The "value" of any $f \in A$ at the point x is given by the silly formula $f(x) = x(f)$. To make applied mathematicians happier, we can assume R to be $\mathbb{R}$, or $\mathbb{C}$... The functor Ξ_A assigns to every ring R the set of R-valued points of the space $\Xi_A: R \mapsto \Xi_A(R) = \mathrm{Hom}_{\mathbb{G}}(A,R)$ (functorial in R).(1)

I am afraid that this still did not convince the hypothetical (?) old-fashioned reader because in any category $\mathbb{C}$ I can fix an object $A \in \mathrm{Ob}\,\mathbb{C}$, and nobody forbids me to call R-valued points of the "Space" Ξ_A the arrows $A \to R$; sure! but I would like to have local coordinates; any Ξ_A should be embeddable in some affine space $\mathbb{E}^I = \Xi_{k[T_i]_{i\in I}}$ ($\Rightarrow$ existence of a surjective map $R \to A$). But if we want to have coordinates, they should be elements of a ring at every point... To be brief: *we are going to use specific properties of the category* $\mathbb{G}$, both in II-5 (recovery of A from Ξ_A) and in §6 as well: the constructions of the fundamental inverse limits: Cartesian products, fiber products, intersections,

(1) If we select a set of generators $\{r_i\}_{i\in I}$ in R , x has coordinates $r_i(x) = x(r_i)$ (silly formula) in the affine space $\mathbb{E}^I$. GROTHENDIECK does not like to assume I to be finite for technical reasons, but he is tolerant: If $I = [1,2]$ or $[1,2,3]$ we might discover that Ξ_A has points with coordinates in R^2 or R^3...

kernels. Every time we shall replace the category Aff with its dual by reversing arrows and then we shall transform the problem in Aff into a problem in A. This problem in A involves BOURBAKI's commutative algel Then we dualize the solutions coming back to Aff_k...; we use geometric intuition to do that... The first instance of this construction is to recover the ring A in terms of the affine algebraic space $\mathfrak{X}_A$. GROTHENDIECK, will convince us that the set of morphisms $\mathrm{Hom}_{\mathrm{Aff}}(\mathfrak{X}_A, \mathbb{E}^1)$ of the space $\mathfrak{X}_A$ in the affine line $\mathbb{E}^1 = \mathfrak{X}_{k[T]}$ has a natural ring structure and moreover there is a canonical isomorphism

$$(4.2) \qquad \mathrm{Hom}_{\mathrm{Aff}_k}(\mathfrak{X}_A, \mathbb{E}^1) \cong A^{(1)}$$

Let us follow GROTHENDIECK's own words:...

5. RECOVERY OF A FROM $\mathfrak{X}_A$..... and we want to give an interpretation of the ring A in terms of the affine algebraic space $\mathfrak{X}_A$ over k, i.e. in terms of the representable functor $\mathfrak{X}_A$ $(A \in \mathrm{Ob}\,\mathfrak{A}_k)$ defined by

$$(5.1) \qquad k' \mapsto \mathfrak{X}_A(k') = \mathrm{Hom}_{k\text{-alg}}(A, k')$$

We can do it, actually, in a very intrinsic and formal way! Let us look into the set

$$(5.2) \qquad \mathrm{Hom}_{\mathrm{Aff}_k}(\mathfrak{X}_A, \mathbb{E}^1) \qquad (\mathbb{E}^1 = \mathfrak{X}_{k[T]} \text{ the affine line over } k)$$

(1) This is the closest thing we can imagine to the classical case involving the classical restriction cf. [0,1]: $A = k[\xi_1, \ldots, \xi_n]$, Nil $A = 0$, k algebraically closed, A becomes the ring of polynomial functions of the algebraic variety Max A, but polynomial functions are maps Max $A \to \mathbb{E}^1(k)$.... GROTHENDIECK recovered this situation even when A was an arbitrary commutative ring with unit: (A might have no subfield k, Nil A may be $\neq 0$,...). Of course in the absolute case $\mathrm{Aff} = \mathrm{Aff}_{\mathbb{Z}}$ and $\mathbb{E}^1 = \mathfrak{X}_{\mathbb{Z}[T]}$.

of morphisms of $\mathfrak{X}_A$ into the affine line over k. Since $\mathbb{E}^1$ is represented by the polynomial ring in one variable T, we have the set equivalences

$$\text{(5.3)} \qquad \mathrm{Hom}_{\mathrm{Aff}_k}(\mathfrak{X}_A, \mathbb{E}^1) \rightrightarrows \mathrm{Hom}(k[T], A) \rightrightarrows A$$

obtained by arrow-reversal in the two opposite categories, where the last $\rightrightarrows$ is a consequence of the fact that any (k-alg)-homomorphism $u\colon k[T] \to A$ is uniquely determined by $u(T) \in A$ and for any $a \in A$ there exist a unique u such that $u(T) = a$. In other words the underlying set of A is recovered from the set (5.2). But let us recall that <u>$\mathbb{E}^1$ is the forgetful functor assigning to every k-algebra k' its underlying set</u>. But k' has a k-algebra structure. Thus our functor $\mathbb{E}^1$ has a k-algebra structure[(1)].

6. FIBER PRODUCTS AND KERNELS IN AFF_k.

SUMMARY. The inverse limits of any type in Aff_k correspond bijectively to the dual direct limits in the opposite category $\mathfrak{a}_k$ (cf. §3).[(2)]
Because of the reduction of §2 it would be sufficient to establish either

1) <u>The existence of arbitrary products in Aff_k</u> (including <u>the existence of a final object</u> $\mathfrak{X}_k$ <u>in</u> Aff_k <u>and fiber products</u> $X \times_S Y$ <u>for any pair</u> $X \to S$, $Y \to S$ <u>in</u> Aff_k).

or

2) <u>The existence of arbitrary products in Aff_k</u> <u>and kernels of arbitrary pairs of morphisms</u> $\mathfrak{X}_A \rightrightarrows \mathfrak{X}_B$.

By duality we need to check the existence of arbitrary coproducts (sums, amalgamated sums) in $\mathfrak{a}_k$ and cokernels of arbitrary pairs of arrows $A \overset{u}{\underset{v}{\rightrightarrows}} B$. Both type of verifications are trivial for a reader with an adequate background in commutative algebra. The finite sums can always be reduced to the case of <u>two</u> k-algebras A, B. Their categorical sum in $\mathfrak{a}_k$ is the

(1) GROTHENDIECK quoted his previous lecture cf. AAG. In the meantime the reader can look at DIEUDONNÉ, Advances, II, page 46 or EGA-Springer, §1.4.4 page 41.

(2) The embedding $i\colon \mathcal{C} \hookrightarrow \hat{\mathcal{C}}$ (cf. Ch. I, §1) identifies any $\mathfrak{X}_A \in \mathrm{Ob}\,\mathcal{C}$ with a contravariant functor from $\mathcal{C}$ to sets or equivalently with a covariant functor from $\mathfrak{a}_k$ to sets: $k' \mapsto \mathrm{Hom}_{k\text{-alg}}(A, k')$, $\forall k' \in \mathfrak{a}_k$. We know (cf. §2) as a purely categorical fact that i is compatible with inverse limits (cf. §2). In other words: if we have a diagram in $\mathcal{C}$ its inverse limit $\varprojlim i(X_\alpha)$ always exists in $\hat{\mathcal{C}}$; we say that it belongs to $\mathcal{C}$ iff $\varprojlim i(X_\alpha)$ <u>is representable</u>. Then the $\varprojlim$ in $\mathcal{C}$ is defined up to isomorphism as anyone of the representing objects of $\mathcal{C}$. Then the important statement made is that if we have any diagram in $\mathcal{C}$ and we take the inverse limit in $\hat{\mathcal{C}}$, <u>this inverse limit is always representable</u>. Dually, <u>direct limits of any type always exist in the category of k-algebras</u>.

tensor product $A \otimes_k B$. In general it is possible to define $\bigotimes_{\alpha \in I} A_\alpha$ for an <u>arbitrary</u> family of k-algebras. On the other hand coker $(A \overset{u}{\underset{v}{\leftleftarrows}} B)$ is given by the canonical surjective (k-alg.) homomorphisms $A \to A/\mathfrak{J}$ where

$\mathfrak{J}$ = {ideal of A generated by the set $u(b) - v(b)$, $\forall b \in B$}

Let $\{X_\alpha\}_{\alpha \in I}$ be a family of affine algebraic spaces $\mathfrak{X}_\alpha$ ($\in \mathrm{Ob}\,\mathcal{C}$, $\forall \alpha$) indexed by a set I (not necessarily finite). Then $\prod_{\alpha \in I} \mathfrak{X}_\alpha$ can always be defined (up to isomorphism) (cf. §1) as an object of $\mathcal{C}$. This is equivalent to saying,

<u>In the category</u> $\mathcal{G}_k$ <u>of k-algebras, the sum of an arbitrary family of objects always exists</u>. Precisely: It is the tensor product $\bigotimes_{\alpha \in I} A_\alpha$. In other words:

Let $\mathfrak{X}_\alpha = \mathfrak{X}_{A_\alpha}$ ($\forall \alpha \in I$), i.e. $\mathfrak{X}_\alpha$ is the affine algebraic space represented by the k-algebra $A_\alpha \in \mathrm{Ob}\,\mathcal{G}_k$. Then $\prod_{\alpha \in I} \mathfrak{X}_\alpha$ <u>is represented by the tensor product</u>:

(6.1)
$$\prod_{\alpha \in I} \mathfrak{X}_{A_\alpha} = \mathfrak{X}_A \Leftrightarrow A = \bigotimes_{k,\ \alpha \in I} A_\alpha$$

<u>because A is the categorical sum of the</u> A_α (cf. §3). The verification of this property is just a rephrasing of the well-known universal property of $\otimes_k$ [1].

In order to show that A is a categorical sum we shall show that the injections (= "coprojections, cf. §3) $i_\alpha : A_\alpha \to A$ ($\forall \alpha \in I$) given by

---

[1] Cf. BOURBAKI, <u>Linear algebra</u>, §3, page 76.

(6.2) $$i_\alpha(a) = 1\otimes\ldots\otimes 1\otimes \overset{\alpha}{a}\otimes\ldots\otimes 1 \qquad (\forall a \in A)$$

(with a factor 1 for every index $\beta \neq \alpha$), have the well known universal property of the sum (cf. §3).

Let us check first the sum property for two factors: $A'\otimes_k A''$ $(A',A'' \in \mathrm{Ob}\,\mathcal{C})$. Then the injections are defined by: $a'\mapsto a'\otimes_k 1$ $a''\mapsto 1\otimes_k a''$

Finally when the set of indices I is infinite, we observe that the family of finite subsets $I'\subset I$ form a filtering direct system and that for every finite $I'\subset I$ we can define $A(I) = \bigotimes_{k\,\alpha\in I'} A_\alpha$. Besides if $I'\subset I''\subset I$ (I'' finite) there is an induced k-algebra homomorphism $A(I')\to A(I'')$. Then we can define $A(I) = \varinjlim A(I')$. In fact the direct limit[(1)] of the underlying sets exists and the I' finite $\subset I$ k-algebra structure carries over.

The existence of fiber products (cf. §1) $X\times_S Y$ for any pair of morphisms $X\xrightarrow{f}S$, $Y\xrightarrow{g}S$ in Aff_k is the same, as we know, as the usual product of these two objects in the category Aff_k/S. This category Aff_k/S is opposite to the category $\mathcal{C}_A$ where A is the k-algebra representing S. In fact, let us assume

$$X = \mathfrak{X}_B \quad Y = \mathfrak{X}_C \quad S = \mathfrak{X}_A$$

Then we have k-algebra homomorphisms (going in the opposite direction as f,g) $A\xrightarrow{f^*}B$, $A\xrightarrow{g^*}C$ compatible with the k-algebra structure, i.e. defining commutative diagrams

(6.3)
$$\begin{tikzcd} A \arrow[r] & B \\ k \arrow[u] \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] & C \\ k \arrow[u] \arrow[ur] & \end{tikzcd}$$

(1) A short introduction on Direct limits for algebras can be found in Exercise 14 of ATIYAH-MACDONALD book pages 32, 33.

(where the vertical arrows are the structure morphisms of A as a k-algebra (cf. Ch. I, §2). As a consequence B,C become A-algebras, $X \times_S Y$ exists and is represented by the tensor product $B \otimes_A C$. We have:

<u>The fiber product</u> $X \times_S Y$ <u>of</u> $X \xrightarrow{f} S$, $Y \xrightarrow{g} S$ <u>exists and is equal to</u> $\mathfrak{X}_D$, <u>where</u> $D = B \otimes_A C$ <u>is the A-algebra deduced from the two algebras,</u> B,C <u>defined by</u> f^*, g^*.[1]

The two previous verifications and the first reduction of §2 proves the existence of arbitrary inverse limits in Aff_k.

An alternative way of proving the previous statement is to show the existence of kernels $\mathfrak{X}_A \rightrightarrows \mathfrak{X}_B$ of arbitrary pairs of arrows $A \overset{u}{\underset{v}{\leftleftarrows}} B$ in the category of k-algebras, i.e. we want to prove the existence of a cokernel $(A \overset{u}{\underset{v}{\leftleftarrows}} B)$ in $\mathfrak{A}_k$, i.e. we want to construct a universal arrow $A \xrightarrow{w} k'$ $(k' \in \mathrm{Ob}\,\mathfrak{A}_k)$ making commutative the diagram

$$k' \xleftarrow{w} A \overset{u}{\underset{v}{\leftleftarrows}} B \tag{6.4}$$

in the category $\mathfrak{A}_k$, ($\Leftrightarrow$ $w \circ u = w \circ v$). This means that for every $b \in B$ $u(b) = v(b)$ $\Rightarrow$ $w \circ (u(b) - v(b)) = 0$ for every $b \in B$. In other words:

$$u(b) - v(b) \in \ker w \qquad \forall b \in B \tag{6.5}$$

Conversely any w of $\mathfrak{A}_k$ satisfying (6.5) makes commutative (6.4). But (6.5) implies $\ker w \supset \mathcal{J} = \{$ideal of A generated by all the differences $u(b) - v(b)$, $\forall b \in B\}$. Then it is well known that $A/\mathcal{J}$ together with the

[1] The reader can easily check, just by reversing arrows, how the fiber product diagram (on the left) comes from the tensor product diagram (on the right).

$$\begin{tikzcd} & Z \arrow[d] & \\ & X \times_S Y \arrow[dl] \arrow[dr] \arrow[dd] & \\ X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd} \quad \leftrightarrow \quad \begin{tikzcd} & Z & \\ & B \otimes_A C \arrow[u, dashed] & \\ B \arrow[ur] & & C \arrow[ul] \\ & A \arrow[ul, "f^*"] \arrow[ur, "g^*"'] & \end{tikzcd}$$

canonical surjective homomorphism $A \xrightarrow{p} A/\mathcal{J}$ is the cokernel that we are looking for, i.e. for every w making commutative (6.4) we can fill dotted arrow below making commutative the diagram

(6.6)
$$\begin{tikzcd} A/\mathcal{J} \arrow[drr, dotted] & & A \arrow[ll, "p"'] \arrow[d, "w"] & B \arrow[l, shift left, "u"'] \arrow[l, shift right] \\ & & k' & \end{tikzcd}$$

7. FILTERING INVERSE LIMITS. DIGRESSION ON NOETHERIAN RINGS....let us give a last example of inverse limits in $C = \mathrm{Aff}_k$. Let us a filtering inverse system of objects $\{X_i\}_{i \in I}$ ($X_i \in \mathrm{Ob}\,C$, $\forall i \in I$) indexed by a filtering ordered set I (or a filtered category). Of course the inverse limit $X = \varprojlim X_i$ correspond to a filtering direct limit in the category $\mathcal{G}_k$ of k-algebras. We are going to check that this is one example where the pure set-theoretical constructions carry over the k-algebra structure, i.e. the underlying set of $A = \varinjlim_{i \in I} A_i$ is the direct limit of the underlying sets of the A_i ($\forall i \in I$) . But, because of the filtering the structure of k-algebra is preserved...

Technically the previous construction is used in algebraic geometry to reduce the case of an affine algebraic space $\mathfrak{X}_A$ over k represented by an arbitrary k-algebra A to the case of finitely generated k-algebras, namely $\mathfrak{X}_A$ is the inverse limit of a filtering inverse system of $\mathfrak{X}_{A_i}$, where the A_i are the finitely generated subalgebras of A ; more precisely: A is obtained as a filtering union of k-algebras which are finitely generated over k , i.e. if A', A'' are two finitely generated subalgebras of A their union is also finitely generated. Therefore, by duality the affine algebraic space $\mathfrak{X}_A$ over k is the filtering inverse system $\mathfrak{X}_{A_i}$, where A_i runs through the family of finitely gener k-algebras.

In many situations statements which are first proved in the finite type case can be carried over to the general case by a limit process.

Let us now make a little comment about the $\mathfrak{X}_A$ of finite type over k!:

An algebra A over k is <u>of finite type over k</u> [= finitely generated over k] if and only if $A \cong k[T_1,T_2,\ldots,T_n]/\mathfrak{J}$, i.e. iff A is isomorphic to a quotient of a polynomial ring with coefficients in k in finitely many variables $T_1,T_2,\ldots,T_n$. To give such an isomorphism is the same as to give a set of n generators of A as a k-algebra. In other words A is described by a polynomial ring together with a bunch of equations $[f(T_1,T_2,\ldots,T_n) = 0,\ f \in \mathfrak{J}]$ and it is sufficient to take any family of generators of this ideal $\mathfrak{J}$. In order that the datum of A over k could be considered altogether as a datum of finite type it is convenient that $\mathfrak{J}$ has a finite number of generators. So, we will say that A <u>is of finite presentation if</u> $A = P_I/\mathfrak{J}$ <u>with I finite and $\mathfrak{J}$ finitely generated.</u>[1]

There is another theorem of HILBERT [the <u>Basissatz</u>] which says that in many cases every ideal in such a polynomial ring $k[T_1,T_2,\ldots,T_n]$ is finitely generated, in other words the algebras of finite type are already of finite presentation.

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[1] It can be easily shown as an exercise that if $\mathfrak{J}$ is finitely generated for the presentation $A = P_I/\mathfrak{J}$, the same happens for any other presentation $A = P_{I'}/\mathfrak{J}'$ (I' finite), i.e. <u>$\mathfrak{J}'$ must be also finitely generated.</u>

Precisely:

If k is a field (in the classical case $k = \mathbb{R}$ or $\mathbb{C}$) then any ideal of a polynomial ring over k with finitely many indeterminates is finitely generated.

A ring in which every ideal is finitely generated is called a Noetherian ring. This means that the Basissatz of HILBERT can be expressed by saying that any polynomial ring in finitely many variables over a field is Noetherian.(1)

Of course a field k is trivially a Noetherian ring. The fact that $k[T]$ is Noetherian comes from the fact that $k[T]$ is a principal ideal domain.(2) A better way of expressing the Basissatz is the following:

If k is already a Noetherian ring then $k[T]$ is also Noetherian. (cf. BOURBAKI, or ZARISKI-SAMUEL.)

Then, by induction we see that if k is a Noetherian ring a finitely generated algebra over k is the same as an algebra of finite presentation.

In other words, this means that for every closed algebraic subspace $X \hookrightarrow \mathbb{E}^n$ represented by $A = k[T_1, T_2, \ldots, T_n]/\mathfrak{J}$, X can be described by a finite number of equations.(3)

8. DIRECT LIMITS IN AFF_k.

GROTHENDIECK's SUMMARY:...regarding direct limits $\varinjlim X_i$ in $C = Aff_k$ we know that they correspond to inverse limits $\varprojlim A_i$ $\lceil X_i = \Sigma_{A_i}, \forall i \in \underline{I} \rfloor$ in the opposite category $\mathcal{C}_k \lceil A_i \in \mathrm{Ob}\, \mathcal{C}_k, \forall i \in \underline{I} \rfloor$. These inverse limits always exist in $\mathcal{C}_k$, since the set-theoretic

(1) Equivalently (after Emmy NOETHER) the ascending chain conditions (a.c.c.) for ideals hold: any increasing sequence of ideals $\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \ldots \subset \mathfrak{a}_n \subset \ldots$ of a Noetherian ring A is stationary $\Leftrightarrow \exists m$, such that $\mathfrak{a}_m = \mathfrak{a}_{m+1} = \ldots$ or any strictly ascending chain of ideals $\mathfrak{a}_1 \subsetneq \mathfrak{a}_2 \subsetneq \ldots$ is finite.

(2) But for $k[T_1, T_2]$ it is more difficult to see...

(3) But GROTHENDIECK expressed again his reluctance to unnecessary restrictions. The canonical choice for generators of $\mathfrak{J}$ is $\mathfrak{J}$ itself which is rarely finite!

limit of the underlying sets of the A_i always exist and the algebraic structure carries over. The trouble arises when we want to interpret them in the opposite category $\mathcal{C}$, i.e. the dual direct limits in $\mathcal{C}$ are not too reasonable! For instance, passage to quotients in Aff_k gives pathological results because it would be not the same as passage to quotients in the more general category of schemes (cf. Vol. II).(1)

There is just one operation, which is always reasonable: the finite sum operation $\coprod_{i\in I} X_i$ [$I = \{1,2,\dots,n\}$ finite index set] in the category $\mathcal{C}$ of affine algebraic spaces over k that we are going to describe in detail below.

Let $X_i = \mathfrak{X}_{A_i} \in \mathrm{Ob}\,\mathcal{C}$, $\forall i \in I$ be a <u>finite family</u> of affine algebraic spaces over k represented by the k-algebras A_i ($\in \mathfrak{A}_k$, $\forall_i \in I$). We are going to prove the existence of the <u>categorical sum</u> (= <u>coproduct</u>, cf. § 3) $\coprod_{i\in I} X_i$, equivalent to the existence of finite products in the opposite category $\mathfrak{A}_k$. In fact <u>the product</u> $A = \prod_{i\in I} A_i$ <u>of these finitely many k-algebras is the categorical product and we have</u>

$$\coprod_{i\in I} \mathfrak{X}_{A_i} = \mathfrak{X}_A \Leftrightarrow A = \prod_{i\in I} A_i \tag{8.1}$$

i.e. <u>the sum of the</u> n <u>k-algebras</u> A_i <u>is represented by the product k-algebra</u> <u>A</u>.

[Let us prove that $A = \prod_{i\in I} A_i$ is the categorical product of the n k-algebras A_i: First of all an element of A is an ordered n-tuple $(a_1,a_2,\dots,a_n)$ ($a_i \in A_i$, $i = 1,2,\dots,n$) with the sum and product of two elements defined elementwise, the unit element of A being $(1,1,\dots,1)$ and the structure morphism $k \to A$ given by $\lambda(a_1,a_2,\dots,a_n) = (\lambda a_1,\lambda a_2,\dots,\lambda a_n)$ $(\forall\lambda \in k)$. We have a decomposition of 1 as a sum of orthogonal idempotents:

(1) This comes from the fact that the yoga we used so far [embedding $i: \mathcal{C} \to \hat{\mathcal{C}}$ of the category $\mathcal{C}$ of affine algebraic spaces over k in the category of contravariant functors from $\mathcal{C}$ to sets, or equivalently in the category of covariant functors from $\mathfrak{A}_k$ to sets] is compatible with inverse limits but it is not compatible at all with direct limits.

(8.2) $\quad 1 = \sum_{i \in I} e_i \quad e_i^2 = e_i = (0,0,\ldots,0,\overset{\downarrow}{1},0,\ldots,0)_{(i)} \quad e_i e_j = 0 \quad i \neq j$.

There are n projections $p_i : A \to A/A(1 - e_i) = A_i (\forall \in I)$ where $A(1 - e_i)$ denotes the ideal of A generated by $1 - e_i$. For any $R \in G_k$ we have a bijection:

(8.3)
$$\mathrm{Hom}_{G_k}(R,A) \rightrightarrows \prod_{i \in I} \mathrm{Hom}_{G_k}(R,A_i)$$

REMARK. The categorical sum of a family $\{S_i\}_{i \in I}$ of sets in the category of sets is the disjoint union $\coprod_{i \in I} S_i$. The natural question arises if for a variable $k' \in \mathrm{Ob}\, G_k$ the set $\Xi_A(k')$ is the disjoint union of the sets $\Xi_{A_i}(k')$ $(i \in I)$. It is easy to construct counterexamples in the framework of classical algebraic geometry.(2) The analysis of this naive assumption lead GROTHENDIECK to study the notion of connectedness in the category Aff_k....

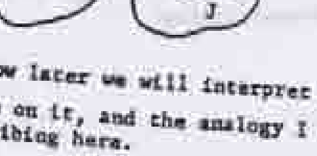
...now we want to see how the k'-valued points of $\Xi_A(k' \in G_k)$ look like. The previous surjective maps $A \to A_i$ correspond to injective maps $\Xi_{A_i} \to \Xi_A$ (written also $X_i \to X$ after simplifying the notations $X_i = \Xi_{A_i}$, $X = \Xi_A$). For every k' we have an injective map $X_i(k') \to X(k)$ defining a map of the disjoint union $\coprod_{i \in I} X_i(k')$ into $X(k')$:

(8.4)
$$\coprod_{i \in I} X_i(k') \to X(k')$$

The question is to check whether (4) is an isomorphism. (Cf. the footnote (2) below to show that it is not necessarily so). We shall now check an ANALOGY WITH TOPOLOGICAL SPACES (3). We have a direct sum $X = \coprod_{i \in I} X_i$

(1) There are also set-theoretic injections $a_i \mapsto (0,0,\ldots,\overset{\downarrow}{a_i},\ldots,0)$ but they are not G_k-morphisms because the unit of A_i goes to the idempotent e_i which is not the unit of A for $n > 1$.

(2) If X,Y are irreducible affine algebraic varieties over the algebraically closed field k represented by the integral domains A,B $X \amalg Y$ represented by $A \times B$ is the disjoint union of the k-valued points of X and Y. If $I \subset X$, $J \subset Y$ are irreducible subvarieties $I \amalg J$ is a k'-valued point of $X \amalg Y$ where k' is the restriction k-algebra of $A \times B$ to $I \amalg J$.



(3) Anyhow later we will interpret every $X_i \in \mathrm{Ob}\, \mathrm{Aff}_k$ as a topological space with a sheaf of rings on it, and the analogy I am going to make is quite relevant to the situation we are describing here.

of topological spaces $X_i \in \mathrm{Ob}\,\mathfrak{T}$ in the category of topological spaces. X a topological space which admits the X_i as open topological subspaces which are mutually disjoint and cover X. Thus set-theoretically X is the disjoint union of the X_i and the topology of X is the unique one in which every X_i is open and the induced topology on each X_i is the given one. In other words U $(\subset X)$ is open in X iff $U \cap X_i$ is open in X_i for every $i \in I$.

Now, on the other hand if we look at an arbitrary topological space Z we have a map

$$(8.5) \qquad \coprod_{i \in I} X_i(Z) \to X(Z)^{(1)}$$

I wonder whether (8.5) is bijective. This means that any continuous map $Z \to X$ can be factored in just one way through one of the X_i, i.e. there exist one and only one index $i \in I$ such that the diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \\ X_i \arrow[ur] & \end{tikzcd}$$

commutes.

Now <u>this is certainly true if Z is connected</u>. But if Z contains several connected components we can map two of them in different X_i's and the previous property would be false.[2]

Coming back to Aff_k we should expect that (5) should be bijective in this category iff Z is connected. <u>This leads us to make explicit,</u>

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[1] The notation $X(Z)$ is short for $\mathrm{Hom}_{\mathfrak{T}}(Z,X)$, i.e. it is the set of continuous maps $Z \to X$.

[2] Cf. this statement with the classical algebraic-geometric situation described in the footnote (2) of page II-8-3.

<u>the notion of connectedness in the category</u> Aff_k .

CATEGORICAL DEFINITION OF CONNECTEDNESS. A topological space X is connected iff any sum decomposition $X = X' \amalg X''$ $(X' \cap X'' = \emptyset)$ implies that one of the summands X' (or X'') is the empty set $\emptyset$. Recalling that $\emptyset$ is an initial object in Sets, we have the following generalization to an arbitrary category $\mathcal{C}$.

An object $X \in \mathrm{Ob}\,\mathcal{C}$ is said to be <u>connected</u> iff whenever X is written as a sum of two objects $X', X'' \in \mathrm{Ob}\,\mathcal{C}$:

$$X = X' \amalg X''$$

this implies that X' (or X'') is an initial object of $\mathcal{C}$.[1]

An initial object in the category Aff_k is called, by analogy, <u>an empty affine algebraic space over k</u> . It should be represented by a final object in the category $\mathfrak{a}_k$. The zero ring 0 is such an object (i.e. the product of an empty family of k-algebras).[2]

The initial object in Aff_k , $\mathfrak{X}_0$ is the functor assigning to every k-algebra k' the set of morphisms $0 \to k'$. This set is empty if $k' \neq 0$ (i.e. if $1 \neq 0$ in k')[3] ..... and it has just one element, the identity for $k' = 0$. $\mathfrak{X}_0$ is called <u>the empty algebraic space</u>.[4]

Now we can define connectedness in the category Aff_k . $X = \mathfrak{X}_A$ is called <u>connected</u> iff any sum decomposition $X = X' \amalg Y''$ implies that either X' or X'' is the empty algebraic space, i.e. iff any product decomposition

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(1) We recall that $\emptyset$ is the initial object in the category $\mathfrak{T}$ of topological spaces as well.

(2) I.e. for every k-algebra k' there is one and only one homomorphism from k' to 0 .

(3) Remember that an $f\colon 0 \to k'$ in $\mathfrak{a}_k$ should have the property $f(0) = 0$, $f(1) = 1$ and this is impossible if $1 \neq 0$ in k !

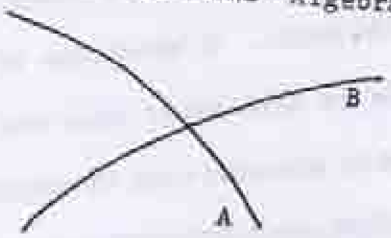
(4) However $\mathfrak{X}_0$ is not the empty functor. For any $A \in \mathrm{Ob}\,\mathfrak{a}_k$, $\mathfrak{X}_A$ is never empty since $\mathfrak{X}_A(A)$ contains always the identity $1_A\colon A \to A$.

$A = A' \times A''$ implies that either A' or A'' is the zero k-algebra. Wl does this mean in terms of commutative algebra? We know that to give a decomposition of a ring A' as a product of two rings A', A'' is equi to define a sum $1 = e' + e''$ of the unit element $1 \in A$ as a sum of tı orthogonal idempotents[1] $e'^2 = e'$, $e''^2 = e''$, $e'e'' = 0$[2].

We shall say that $A \in \mathrm{Ob}\,\mathfrak{G}_k$ is connected ($\Leftrightarrow$ $\mathfrak{X}_A$ is connected) b abuse of language iff every decomposition $1 = e' + e''$, $e'^2 = e'$, $e''^2 =$, $e'e'' = 0$ is trivial, i.e. either $e' = 0$ or $e'' = 0$. (Every idempoter is either equal to zero or to one).

EXAMPLE. Every integral domain A is connected. Otherwise if $e' \neq 0$ is an idempotent $1 - e'$ is also an idempotent and $e'(1 - e') =$ $\Rightarrow 1 - e' = 0$.

The previous example shows again the well-known relationship (coming from "old time" Algebraic Geometry) between integral domains and irreducibi



If an "algebraic variety" X can be decomposed (non trivially) as a union $X = A \cup B$ of two proper subvarieties it is possible to construct polynomial functions f, g with $f(g)$ vanishing identically in $A(B)$ but not in $B(A)$; the product fg vanishes in X but $f \neq 0$, $g \neq 0$. This "reducibility" implies the existence of divisors of zero. In Ch. III, §5 we shall see how the old notion of irreducibility is reall a general topological notion, particularly expressive in the ZARISKI topology.

(1) Frequently called also projectors because of well-known geometrical meanings.

(2) Similarly we can see as before that any finite decomposition $A = A_1 \times A_2 \times \dots \times A_n$ is equivalent to a decomposition in mutually orthogonal idempotents e_i whose sum is equal to 1. Conversely for any such decomposition we recover the A_i by $A_i = A/A(1 - e_i)$ $(i = 1, 2, \dots, n)$.

CHAPTER III

AFFINE SCHEMES

Ch. III is independent of Ch. II. In EGA, I, Ch. I, §1 page 79 GROTHENDIECK starts Algebraic Geometry with the definition of affine schemes (which do not require anything from Ch. I as a logical prerequisite). Ch. I however, provides a very good geometric motivation, starting with systems of polynomial equations.

The most famous fact of GROTHENDIECK's work is the replacement of the algebraic varieties by the schemes as the main subject of study of Algebraic Geometry. The simplest types of schemes are the affine schemes. They are building blocks of all the schemes (Cf. Vol. II, Ch. V).

Affine schemes play an analogous rôle to affine algebraic varieties. Actually GROTHENDIECK's definition was inspired by SERRE's sheaf theoretic definition of algebraic varieties in the FAC paper. A knowledge of FAC is not logically indispensable and to include it here would take too much space. Since this FAC material was the main prerequisite of another Buffalo course by GROTHENDIECK(devoted to the RIEMANN-ROCH-GROTHENDIECK theorem), I will include a summary of such theory at the beginning of Vol. II.

....The geometric language used previously is associated with the representable functor $X: C_k \to$ Sets. $X(k') = V_A(k') = \mathrm{Hom}_{k\text{-}alg}(A,k')$ is the set of k' valued points $(k' \in \mathrm{Ob}\, C_k)$ (Cf. Ch. I). We shall now use another geometrical language, which associates to the functor $X(= V_A)$ certain topological spaces which have an extra-structure, the so-called affine schemes. One reason we do this is that affine schemas allow us to define more general algebraic objects, the so-called schemes. Schemes are obtained by gluing together certain 'affine' pieces (the affine schemes). This process is parallel to the construction of projective algebraic varieties in terms of certain affine pieces (affine algebraic varieties). These projective algebraic varieties (for instance the complex ones) refuse to be embeddable in an affine space. Besides there are other ways of gluing together affine algebraic varieties or affine schemes. We need to develope a process to glue these different pieces together. One of the most intuitive ways of doing so employs the theory of sheaves. As a consequence, we are going to study topological spaces X endowed with an additional structure, a sheaf $\mathcal{O}_X$.

⌈SUMMARY. The functor affine algebraic space $\mathfrak{X}$ over k, or equivalently the k-algebra A which represents $\mathfrak{X}$ ($\mathfrak{X} = V_A$) has the serious inconvenience of not attaching to A a single geometric object but a family of geometric objects $V_A(k')$ varying with k'. We are going to attach to A (and thus to $\mathfrak{X}$) a <u>geometric object</u>, the <u>affine scheme</u> (Spec A,$\tilde{A}$).(1) We shall study separately the topological space X = Spec A defined by a certain spectral topology (cf. §2 p59), naturally described by the radical ideals $\mathfrak{a} = \sqrt{\mathfrak{a}}$ of A : F(⊂ X) <u>is closed iff there exists a radical ideal</u> $\mathfrak{a} = \sqrt{\mathfrak{a}}$ such that $x \in F \Leftrightarrow \mathfrak{p}_x \supset \mathfrak{a}$(2). As a consequence, Spec A = Spec(A/Nil A). Thus one cannot recover A from X unless we can restrict ourselves to <u>reduced algebras</u>, i.e. to the case Nil A = 0 (i.e. zero is the only nilpotent element of A).

In order to recover A we need to define <u>the structure sheaf</u> $\mathcal{O}_X = \tilde{A}$ <u>of</u> X. Then $(X,\mathcal{O}_X)$ gives back A because A becomes isomorphic with the k-algebra of global sections on X, i.e. with the 0^{th} cohomology group, $A \cong \Gamma(X,\mathcal{O}_X) = H^0(X;\mathcal{O}_X)$. An interesting side remark is that k does not play any privileged role in the construction: we only used the structure of a commutative ring with unit A. Thus the functor $A \mapsto (\mathrm{Spec}\,A,\tilde{A})$ shows that the affine schemes (Spec A,$\tilde{A}$) attached to A can be identified with the objects of the category $\mathfrak{C}^o$ opposite to the category $\mathfrak{C}$ of commutative rings with unit and $A \xrightarrow{f} B$ goes to $(\mathrm{Spec}\,B, \tilde{B}) \xrightarrow{f^*} (\mathrm{Spec}\,A,\tilde{A})$.⌋

PART I

THE FUNCTOR Spec: $\mathfrak{C} \to \mathfrak{T}$

⌈SUMMARY. In this Part I we study the contravariant functor Spec which goes from the category $\mathfrak{C}$ of commutative rings with units and unit preserving ring homomorphisms to the category $\mathfrak{T}$ of topological spaces and continuous maps.⌋

The set Spec A = {𝔭|𝔭 prime ideal of A} appeared before several times (cf. Ch. I, page 10). Spec A has a natural topology called <u>the spectral topology</u> or also <u>the ZARISKI topology of A</u> ⌈because it is a natural intrinsic version of the topology already studied in Ch. I, §14.⌋ The map

(0.1) $$\mathrm{Spec}\,\varphi\colon \mathrm{Spec}\,B \to \mathrm{Spec}\,A$$

induced by a morphism $\varphi\colon A \to B$ in the category $\mathfrak{C}$ of commutative rings with unit is continuous, thus <u>Spec is a contravariant functor</u> from $\mathfrak{C}$ to <u>the category $\mathfrak{T}$ of topological spaces and continuous maps.</u> ⌋

(1) Cf. Summary of Vol. I . Spec A, the Spectrum of A is the underlying topological space, $\tilde{A}$ is the structure sheaf on Spec A. The couple (Spec A,$\tilde{A}$) is a locally ringed space since $\tilde{A}$ is a sheaf of local rings over Spec A.

(2) Every element of Spec A is written with a double notation $x, \mathfrak{p}_x$ according to the fact that we regard it as a point $x \in$ Spec A or as a prime ideal $\mathfrak{p}_x \subset A$.

§1 provides the link with Ch. I (although a direct and very elementary check suffices to establish this relation.)

1. LOCI OF A. THE SPECTRUM OF A. The <u>loci</u> of A , were introduced in Ch. I, §8 as <u>equivalence classes of geometric points</u>. The quotient set is Spec A that can be defined directly as the set of all prime ideals of A .

We want to make clear why the ground ring k does not play any particular role in the construction of Spec A, Max A , <u>or even later</u> (cf. §2) <u>in order to introduce the so-called spectral topology of</u> A . Let $h: k \to A$ be the structural morphism of A ($h \in G$) as a k-algebra. Let $A \xrightarrow{f} B$ be an arbitrary morphism in G . Then B acquires a well-defined structure as a k-algebra just by taking $f \circ h$ as structural morphism. We are going to assume $k = \mathbb{Z}$, i.e. in the sequel we identify G with $G_{\mathbb{Z}}$.

Let A be a ring homomorphism ($\in G$) . Let $\mathfrak{b}$ be any ideal of B. Then the following properties hold.

1)[1] <u>The inverse image</u> $\varphi^{-1}(\mathfrak{b})$ <u>is an ideal of A</u>

2)[1] <u>If $\mathfrak{b}$ is prime</u> $\varphi^{-1}(\mathfrak{b})$ <u>is also a prime ideal of A</u>

3)[1] In particular <u>if $\mathfrak{m}$ is a maximal ideal of B</u> , $\varphi^{-1}(\mathfrak{m})$ <u>is a prime</u> ideal of A, but <u>$\varphi^{-1}(\mathfrak{m})$ is not necessarily maximal</u> (Cf. counter-example below).

1) We leave the verification to the reader.

The property 2) implies that Spec can be regarded as a contravariant functor: $G \to$ Sets, where Spec φ (cf. 10.1) is defined by $\mathfrak{b} \mapsto \varphi^{-1}(\mathfrak{b})$. Check functor properties! We shall see in § 2 that Spec φ is continuous when we introduce in A and B the spectral topology. In other words:

Spec will be regarded as a contravariant functor from $\mathbb{G}$ to the category $\mathfrak{T}$ of topological spaces and continuous maps.

COUNTER-EXAMPLE for $\operatorname{Max}(A)$. Let $i: \mathbb{Z} \hookrightarrow \mathbb{Q}$ be the natural inclusion of $\mathbb{Z}$ into the field of rational numbers. $\operatorname{Max}(\mathbb{Q}) = \operatorname{Spec}(\mathbb{Q}) = \{0\}$ is a one-point set containing just the zero ideal. $i^{-1}(0)$ is the zero ideal of $\mathbb{Z}$; $0 = i^{-1}(0) \in \operatorname{Spec}(\mathbb{Z})$, i.e. the zero ideal of $\mathbb{Z}$ is prime $\Leftrightarrow$ $\mathbb{Z}$ is an integral domain. But $i^{-1}(0)$ is not maximal: For any prime $p > 1$ we have the inclusion $(0) \subset (p)$.

This functorial misbehavior of A , lead GROTHENDIECK to reject Max A as a kind of intrinsic replacement for the various $\mathfrak{X}(k')$ $(k' \in \operatorname{Ob}\mathbb{G})$ in spite of the fact that if k is an algebraically closed field and $\mathfrak{X}(k)$ is embedded in k^n we can establish a (1-1)-map between the maximal ideals of A and the k-valued points of this affine model.

2. THE SPECTRAL TOPOLOGY.

⌈The material of this § is developed in full detail in BOURBAKI's COMM. ALG., 1290, XXVII Ch. II, Localization, §3, page 124. The original exposition of GROTHENDIECK in EGA, I, IHES has been suppressed in EGA-Springer (Ch. I, §1, 1.1, page 194) where only a few complementary properties are developed in full. EGA-Springer starts with the structure sheaf.[1] In this course GROTHENDIECK considered "first" that A is an algebra over the ground ring k . Actually this hypothesis is not restrictive because every A can be regarded as a $\mathbb{Z}$-algebra by means of the unique structural morphism $\mathbb{Z} \to A$.⌋

SUMMARY. We construct and study the properties of the contravariant functor Spec from the category of k-algebras to the category of topological spaces. A closed set Y of the topological space X = Spec A can be characterized as sets of all solutions of arbitrary systems $S: (f_j(u) = 0)$ but we need to give an intrinsic meaning to $f_j(u)$ in order to show that the vanishing depends only on the loci of $A_{(1)}$ (Cf. I, §8). It is proved that S can be a radical ideal $a = \sqrt{a}$ uniquely determined by Y[2]. GROTHENDIECK removed all the older restric-

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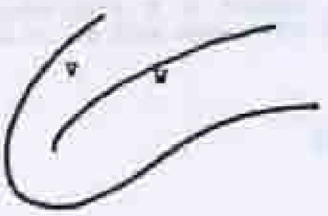
(1) Other quick introductions (without proofs) can be found in MCDONALD, DIEUDONNÉ - Advances I cf. also MUMFORD's notes. Cf. also LANG, ATIYAH-MCDONALD, The elementary approach starting with systems of equations given in the Introduction to EGA-Springer is not used in the body of the book although it appears scattered in EGA. I could not find it elsewhere except in MANIN's-MΓY notes.

(2) ⌈This agrees with the classical case (k a field, A a finitely generated k-algebra without nilpotent elements): If A is the ring of polynomial functions of a variety V and W is a subvariety the ring of polynomial functions on W , the restriction $B = A|W$ is defined by $B = A/a$ $a = \sqrt{a}$ ideal of all functions of V vanishing at W . Essentially, this geometric consideration dictates the choice of the topology.⌋



tions on A (in particular k disappears showing that all the essential properties remain.)
In other words: The family of closed sets in this topology can be described completely in
terms of the radical ideals. As a consequence $X = \mathrm{Spec}(A)$ is homeomorphic with $\mathrm{Spec}(A/\mathrm{Nil}A)$.
A subset $Y \subseteq X$ is closed iff Y is the set of inverse images $\mathrm{Spec}(A/\mathfrak{a})$ for some radical
ideal $\mathfrak{a} = \sqrt{\mathfrak{a}}$ of A by the canonical projection $A \to A/\mathfrak{a}$. (Cf. footnote (*) in p. 79).

Spec(A) is a KOLMOGOROFF space (i.e., it satisfies the T_0-separation axiom, but Spec(A) is not T_1, and (of course!) it is even less T_2 (= HAUSDORFF) except in trivial cases.(*)

Spec(A) is quasi-compact.(**)

For the time being Spec A is just a set:

$$(2.1) \qquad X = \mathrm{Spec}\, A = \{\mathfrak{p} \mid \mathfrak{p} \text{ prime ideals of } A\}$$

We are going to describe the family of closed sets of $X = \mathrm{Spec}\, A$ in a certain topology, called the spectral or ZARISKI topology of X (cf. Ch. I, §

Let V(S) be the subset of X defined by

$$(2.2) \qquad V(S) = \{\text{Set of } \underline{\text{loci}} \text{ of } u \in \mathfrak{X}(k') \mid f_i(u) = 0,\ \forall f_i \in S\} \qquad i \in I$$

where $\mathfrak{X}_A$ denotes the contravariant functor $C_k \to$ Sets represented by A. In other words $\mathfrak{X}$ is the affine algebraic space over k represented by A. $u\colon A \to k'$ is a geometric point of $\mathfrak{X}$, i.e. the k-algebra k' is a field (Cf. Ch. I, §8).

(*) T_0-separation axiom (KOLMOGOROFF): For every pair (x,y) of points of the topological space at least one of them has an open neighborhood which does not contain the other, i.e. $\exists\, U_x$ open $\ni x$, $y \notin U_x$ or $\exists\, U_y$ open $\ni y$, $x \notin U_y$.

T_1-separation axiom (FRÉCHET). For both x,y as before $\exists\, U_x \ni x$, and $y \notin U_x$ and $\exists\, U_y \ni y$, $x \notin U_y$ (both U_x, U_y open). T_1 is equivalent to the property that every one-point set $\{x\}$ is closed. T_2 (HAUSDORFF): $\exists$ disjoint open neighborhoods U_x, U_y open st: $U_x \cap U_y = \emptyset$.

(**) A topological space X is quasi-compact iff any open covering $X = \bigcup_{i \in I} X_i$ (X_i open in X, $\forall\, i \in I$) contains a finite subcovering. X compact $\Leftrightarrow$ X quasi-compact and HAUSDORFF. The topological spaces appearing in this course will not be HAUSDORFF except in trivial cases.

If $f \in A$, $u \in \mathfrak{X}(k')$ $f(u) \in k'$ is defined by

$$f(u) = u(f) \tag{2.3}$$

thus (2.2) makes sense, expressing that u satisfies a "system of equations" $\{f_i\}_{i \in I}$.

On the other hand (2.2) is compatible with the equivalence relation defining the <u>loci</u> of A, i.e. it is independent of the representatives. It depends only on the loci. <u>In other words</u>, if we have a k-algebra homomorphism $k' \to k''$ (necessarily injective since k', k'' are fields) then $f_i(u) = 0 \Rightarrow f_i(v) = 0$ where v is the image of u by the induced map, i.e. since $f_v(u) = u(f_v) = 0$. This means that f_i belongs to the kernel of u. Let $\mathfrak{p} = \ker u \in \operatorname{Spec} A$. We see that <u>$V(S)$ is just the set of all prime ideals $\mathfrak{p}$ containing S</u>:

$$V(S) = \{x \in \operatorname{Spec} A \mid \mathfrak{p}_x \supset S\} \tag{2.4}$$

Of course $\mathfrak{p} \supset \mathfrak{J}_S$ (the ideal generated by S). Thus we have

$$V(S) = V(\mathfrak{J}_S) \tag{2.5}$$

But since a prime ideal $\mathfrak{p}$ is always equal to its radical: $\mathfrak{p} = \sqrt{\mathfrak{p}}$ $V(S) = V(\mathfrak{J}_S)$ does not change if we replace $\mathfrak{J}_S$ by its radical $\sqrt{\mathfrak{J}_S}$ i.e. we have

$$V(S) = V(\mathfrak{J}_S) = V(\sqrt{\mathfrak{J}_S})^{(*)} \tag{2.6}$$

In words: <u>The prime ideals</u> $\mathfrak{p}$ <u>of</u> A <u>containing</u> S <u>are just the</u>

(*) I avoid the notation $\bar{\mathfrak{a}}$ for the radical $\sqrt{\mathfrak{a}}$, because later $\bar{A}$ will denote the structure sheaf.

prime ideals containing the radical $\sqrt{\mathfrak{J}_S}$ of the ideal $\mathfrak{J}_S$ generated by S.

Now we can't go any further, because we can recover $\sqrt{\mathfrak{J}}$ from $V(\sqrt{\mathfrak{J}})$ by the formula:

$$\sqrt{\mathfrak{J}} = \bigcap_{x \in V(\mathfrak{J})} \mathfrak{p}_x \tag{2.7}$$

i.e. because of the property that $\sqrt{\mathfrak{J}}$ is the intersection of all the prime ideals of A containing $\mathfrak{J}$(*).

Therefore we proved the following property:

The sets that can be written in the form $V(S)$ are identical with sets of type $V(\mathfrak{J})$ with $\mathfrak{J} = \sqrt{\mathfrak{J}}$ and $\mathfrak{J}$ is recovered from $V(\mathfrak{J})$ by the formula (2.7).

In other words we checked that the family of sets of Spec A of type $V(S)$ for some S, (called in the sequel algebraic subsets of X), are in (1-1)-correspondence (reversing inclusions) with the radical ideals of A, i.e. if $\mathfrak{J} = \sqrt{\mathfrak{J}}$, $\mathcal{J} = \sqrt{\mathcal{J}}$ and $\mathfrak{J} \subsetneq \mathcal{J} \Rightarrow V(\mathcal{J}) \subsetneq V(\mathfrak{J})$.

Now we are going to see how these algebraic subsets behave with respect to unions and intersections. To do this we are going to write down two formulas (2.8), (2.9). First

$$\bigcap_{i \in I} V(\mathcal{J}_i) = V(\bigcup \mathcal{J}_i) = V(\sum \mathcal{J}_i) \tag{2.8}$$

In other words: The intersection of an arbitrary family of algebraic subsets of Spec(A) is still an algebraic subset of A.

Now, let us look for finite unions! Then we have:

(*) We can see this by "lifting to A" by means of the canonical map $A \to A/\mathfrak{J} = B$ the property that $\mathrm{Nil}\, B$ = {set of all nilpotent elements of B} = intersection of all prime ideals of B.

(2.9) $$V(\prod_{1\le i\le n} \mathcal{J}_i) = \bigcup_{1\le i\le n} V(\mathcal{J}_i)$$

(2.8) is quite evident. (2.9) can be proved as follows: First the embedding $\prod_{1\le i\le n} \mathcal{J}_i \subset \mathcal{J}_i$ leads to the opposite inclusions $V(\prod_{1\le i\le n} \mathcal{J}_i) \supset V(\mathcal{J}_i)$, for every i

(2.10) $$V(\prod_{1\le i\le n} \mathcal{J}_i) \supset \bigcup_{1\le i\le n} V(\mathcal{J}_i)$$

Conversely, we want to prove that $x \in V(\prod_{1\le i\le n} \mathcal{J}_i)$ implies $x \in V(\mathcal{J}_i)$ for some i. Otherwise if $\mathfrak{p}_x \not\supset \mathcal{J}_i$ for $i = 1,2,\ldots,n$ there exists $f_i \in \mathcal{J}_i$ with $f_i \notin \mathfrak{p}_x$. However $f_1 f_2 \ldots f_n \in \mathfrak{p}_x$ in contradiction with the fact that $\mathfrak{p}_x$ is prime.

Obviously, we have

(2.11) $$\emptyset = V(A) \qquad X = V(0) = V(\mathrm{Nil}A)$$

This, together with the previous remarks, tells us that <u>the family of sets defined by $V(\mathfrak{J})$ $(\mathfrak{J} = \sqrt{\mathfrak{J}})$ contains $\emptyset$ and X, and it is stable by arbitrary intersections and by arbitrary finite unions</u>. In other words:

The family $V(\mathfrak{J})$ ($\mathfrak{J} = \sqrt{\mathfrak{J}}$ radical ideal of A) satisfies the axioms for closed sets for a topology on $X = \operatorname{Spec} A$. This topology is called the <u>spectral</u> or <u>ZARISKI topology of X</u> = Spec A. This notation will denote in the sequel, the corresponding topological space.

Let us prove now that <u>X is quasi-compact</u> [1] in other words we want to prove that if $X = \bigcup_{i\in I} U_i$ (U_i open, $\forall i \in I$, I arbitrary), there exists a finite subfamily covering X. Now in terms of closed sets, this property is equivalent to the fact that if $(Y_i)_{i\in I}$ is an arbitrary

[1] Cf. definition in the footnote of page III-2-2.

family of closed sets with an empty intersection: $\bigcap_{i \in J} Y_i = \phi$ there exists a finite subfamily whose intersection is empty already. Let $\mathfrak{J}_i$ be ideals of A, such that $Y_i = V(\mathfrak{J}_i)$. The sum $\Sigma \mathfrak{J}_i = \mathfrak{J}$ of these ideals represent ϕ, i.e. $V(\mathfrak{J})$ = empty. This is equivalent to the fact that $\mathfrak{J}$ is not contained in any prime ideal whatever! Then by KRULL's theorem (which says that any non-zero ring with unit contains at least one maximal ideal and a fortiori a prime ideal) applied to $A/\mathfrak{J}$ we see that $\mathfrak{J}$ is contained in some prime ideal unless $\mathfrak{J} = A$. In other words

(2.12)
$$V(\mathfrak{J}) = \phi \Leftrightarrow \mathfrak{J} = A$$

So $\Sigma \mathfrak{J}_i = A$, as a consequence we can write $1 = \sum_{i \in I} f_i \quad f_i \in \mathfrak{J}_i$ with finitely many $f_i \neq 0$, which implies that A is the sum of finitely many $\mathfrak{J}_i$, i.e. the intersection of finitely many Y_i is empty!

Let us look at the closure $\overline{\{x\}}$ of the one point set $\{x\}$ of X. $\overline{\{x\}}$ is the smallest closed subset of X containing x. Let $Y = V(\mathfrak{J})$. To say that Y contains x is equivalent to saying that $\mathfrak{J} \subset \mathfrak{p}_x$. Thus the smallest closed set containing x, corresponds to the maximal $\mathfrak{J}$ contained in $\mathfrak{p}$ which is $\mathfrak{p}$ itself. As a consequence we have:

(2.13)
$$y \in \overline{\{x\}} \Leftrightarrow \mathfrak{p}_y \supset \mathfrak{p}_x$$

Therefore the specialization relation[1]: $y \in \overline{\{x\}}$ corresponds to the reverse inclusion relation for prime ideals.

COROLLARIES. 1) The one point set $\{x\}$ is closed if the prime ideal $\mathfrak{p}_x$ is maximal.

[1] Let x, y be two points of any topological space X. We say that y is a specialization of x iff $y \in \overline{\{x\}}$. Cf. next §3. Of course in any T_1 space, in particular in HAUSDORFF spaces, y is a specialization of $x \Leftrightarrow y = x$, i.e. the notion becomes trivial!

2) X is a T_1-space ($\Leftrightarrow$ every point of X is closed) iff every prime ideal of X is maximal.

We can show plenty of examples of rings A with nonmaximal prime ideals, thus X is not necessarily a separated space! We shall verify shortly the T_0-axiom. ⌈Cf. footnote (*) in III-2-2⌋.

⌈If $y \in \overline{\{x\}}$, every closed set of X containing x contains also y which is equivalent to: every open neighborhood of $y \in \overline{\{x\}}$ contains also x.

A point g of an irreducible subset Y of a topological space X is called a generic point of Y iff either one of these two equivalent properties hold:

1) $Y = \overline{\{g\}} \Leftrightarrow$ every closed set of X containing g contains also y, $\forall y \in Y$.

2) For every $y \in Y$, any open neighborhood of y contains x;

2) can be rephrased also as follows: every non-empty open set of Y contains the generic point g.

Now we can prove that Spec A is a T_0-space. Let x, y be two different points of Spec A. Let us assume that every open neighborhood of y contains $x (\Leftrightarrow \mathfrak{p}_y \supset \mathfrak{p}_x)$. Then $\mathfrak{p}_x \not\supset \mathfrak{p}_y$ (otherwise $x = y$). Thus $x \notin \overline{\{y\}} \Rightarrow$ some open neighborhood of x does not contain y, q.e.d.

3. THE CANONICAL BASIS $\mathcal{B}$ OF Op(X)(*).

The classical fact that any ZARISKI closed set (old time informal "affine variety": (cf. I, §14) is the intersection of "hypersurfaces": $f(x) = 0$ has a natural analogue in Spec A ($f(x) = 0 \Leftrightarrow f \in \mathfrak{p}_x$). The dual property can be expressed by saying that the family

(*) The blackboard notation for the category of open sets and inclusions in a topological space X was $\mathfrak{G} = \mathfrak{G}(X)$ which is too close to $\mathfrak{O}_X$ (the structure sheaf). I replace it here by $Op = Op(X)$ for open as a translation of his Ouv. for "ouvert".

$\mathfrak{B} = [X_f]_{f \in A}$ of open sets defined by (3.1) is a natural or canonical basis for the topology
of Spec A. $\mathfrak{B}$ is stable for finite intersections. If f is not nilpotent X_f is the
complement of a proper "hypersurface" $V_f = [x \in \text{Spec } A | f(x) = 0]$ as in the classical case.
If f is nilpotent $X_f = \emptyset$. $\mathfrak{B}$ is a natural example of a "site" (GROTHENDIECK topology),
i.e. a topology with covering data in which sheaves can be defined; of footnotes in pages
100 and 101. Some functorial properties needed in Part II in order to define the structure
sheaf of Spec A are established here to avoid a later lengthy digression. They can be left
for a second reading.

The definition of a closed set F on X as $F = V(S)$ $(S \subset A)$ is equivalent to the fact that every closed set F is the intersection of closed sets of type $V(f)$ $(f \in A)$, defined by

(3.1) $$V(f) = \{\mathfrak{p} \in \text{Spec } A | f \in \mathfrak{p}\} = \{\mathfrak{p} \in \text{Spec } A | f(\mathfrak{p}) = 0\}$$

Precisely:

(3.2) $$F = V(S) \Rightarrow F = \bigcap_{f \in S} V(f)$$

Particular important cases of the closed sets $V(f)$ are

(3.3) $$V(1) = \emptyset \qquad V(0) = X$$

More precisely, we have:

(3.4) $$V(f) = X \Rightarrow f \in \mathfrak{p},\ \forall \mathfrak{p} \in \text{Spec } A \Rightarrow f \in \text{Nil} A$$

In other words $V(f) = X$ iff f is nilpotent.

By topological duality we get the following property:

Every open set U of X is the union of open sets of type X_f, where

(3.5) $$X_f = X - V(f) = \{\mathfrak{p} \in \text{Spec } A | f \notin \mathfrak{p}\}$$

Precisely

(3.6) $$F = X - U = V(S) = \bigcap_{f \in S} V(f) \Rightarrow U = \bigcup_{f \in S} X_f$$

The family $[X_f]_{f \in A}$ is called the canonical basis of the Spectral topology on X. It will play an important role in the construction of the structure sheaf in Part II.

The canonical basis is stable by finite intersections, precisely·

(3.7) $$X_{fg} = X_f \cap X_g \qquad \forall (f,g) \in A \times A$$

The inclusion relations between basic open sets can be interpreted in terms of commutative algebra, for instance:

$X_f \subset X_g$ <u>implies the existence of a positive integer</u> n <u>and an element</u> h <u>of</u> A <u>such that</u>

(3.8) $$g^n = hf$$

Proof. $X_f \subset X_g \Rightarrow V(f) \supset V(g) \Rightarrow V(fA) = V(\sqrt{fA}\,) = V(g) \Rightarrow$ (3.8). In particular <u>two elements</u> f,g of A <u>define the same basic open sets</u>: $X_f = X_g$ iff there exist two integers $m > 0$, $n > 0$ such that $f^m \in A_g$, $g^n \in A_f$.

In fact, the reason why we want to emphasize this canonical basis $\mathcal{B} = \{X_f\}_{f \in A}$ is that the exactness properties characterizing the sheaves on X as particular cases of presheaves can be expressed in terms of commutative algebra. To make this more explicit I report GROTHENDIECK's talk, following the tape at this § to avoid too many heavy diagressions in the construction of the structure sheaf $\tilde{A}$ (cf. III, Part II, § 6, 7, 8, ...).

... Before discussing the structure sheaf of rings $\tilde{A}$ in Spec A Cf. Part II of Ch. III I will give the general description of sheaves on the spectrum by using the previous remark that the open sets of type $X_f \in \mathrm{Ob}\,\mathcal{B}$ ($f \in A$) from a basis of the topology stable by finite intersections. Now I recall again that to give a sheaf of sets on an arbitrary topological space X is the same as to define a presheaf satisfying certain exactness conditions; first of all let us recall that for any topological space X we can construct the category Op(X) (cf. footnote (*) of page 95) whose objects are the open sets of Op(X). The set Hom(U,V) of morphisms for two open sets of Op(X) is either empty iff $U \not\subset V$ or it consists of a unique morphism $i_V^U: U \to V$ (the canonical inclusion).

A presheaf P on X with values in a category $\mathbb{C}$ is a contravariant functor $P\colon (Op(X))^{\circ} \to \mathbb{C}$. GROTHENDIECK is particularly interested in considering now the case that $\mathbb{C}$ is the category of Sets.

...A sheaf of sets F on X is a presheaf which associates to every open set U of X a set $F(U)$ and whenever U is covered by open sets U_i: $(i \in I)$

(3.9)
$$U = \bigcup_{i \in I} U_i$$

if we look at the intersections $U_i \cap U_j$ $(i,j \in I)$ by taking $F(U)$ to the product of the restrictions $\rho^{u}_{u_i}\colon F(u) \to F(u_i)$ an this to the twofold product corresponding to the two inclusions of $U_i \cap U_j$ in U_i or U_j we obtain an exact sequence

(3.10)
$$F(U) \to \prod_{i \in I} F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)^{(1)}$$

In other words: To say that the presheaf F is a sheaf is equivalent to say that (3.10) is exact[1] for every choice of the covering (3.9) of U and for every U. Now let us come back to the case $X = \operatorname{Spec} A$. Since $\mathcal{B}$ is a basis stable for intersections (cf. (3.7));it suffices to rephrase the previous general considerations by maps $X_f \mapsto F(X_f)$ $(f \in A,\ X_f \in \mathcal{B})$ and whenever X_f is covered by open sets X_{f_i} $(f_i \in A)$:

(3.11)
$$X_f = \bigcup_{i \in S} X_{f_i} \quad (f, f_i \in A\ ,\ i \in I)$$

---

(1) Exactness means of course that the image of the single arrow is identical with the kernel of the double arrow. In the category of sets, ker(u,v), where $A \overset{u}{\underset{v}{\rightrightarrows}} B$ is a double arrow is defined by

$$\ker(u,v) = \{a \in A \mid u(a) = v(a)\}$$

In general categories the definition is reduced to the set theoretic case, using morphisms, instead of points of objects.

if we look at the intersections $X_{f_i} \cap X_{f_j}$ $(i,j \in I)$ by taking $F(X_f)$ to the product of the restrictions and this to the twofold product corresponding to the two inclusions of $X_{f_i} \cap X_{f_j}$ in X_{f_i} or X_{f_j} we obtain a diagram

(3.12) $$F(X_f) \to \prod_{i \in I} F(X_{f_i}) \rightrightarrows \prod_{i,j} F(X_{f_i f_j})$$

which is exact in an obvious way.

In terms of commutative algebra we shall simplify our notations by calling G the "sheaf" that we want to describe by writing

(3.13) $$G(f) = F(X_f) \quad \forall f \in A$$

Thus to every element f of the ring A we shall associate a set $G(f)$[(1)]. We shall give certain data to this collection of sets $\{G(f)\}_{f \in A}$ and the conditions needed on this data in order to define a "sheaf" G. Explicitly we have the conditions:

a) $f \mapsto G(f) \quad \forall f \in A$

b) For every pair $a, b \in A$ such that $X_f \subset X_g$ I want to define a restriction map:

$$\rho_f^g: G(g) \to G(f)$$

We have to interpret in terms of commutative algebra what means that $X_f \subset X_G$. We saw already (cf. p. 97) that $X_f \subset X_g$ is equivalent to the existence of a positive integer n and an $h \in A$ such that (3.8) holds!

In other words $X_f \subset X_g$ can be expressed in terms of the divisibility relation on the ring A. Moreover we have transitivity, i.e.: $X_f \subset X_g \subset X_h$ imply that the diagram of inclusions:

---

(1) Or $G(f)$ will be a group, a ring, an A-module, etc., according to the category on which G takes values. In the text $F(f)$ continues to be a pure set.

(3.14)

$$\begin{tikzcd} G(h) & G(g) \arrow[l, hook'] \arrow[d] \\ & G(f) \arrow[ul] \end{tikzcd}$$

is commutative. So we have a "presheaf in the category of basic open sets X_f" which can be interpreted as a presheaf in the category A [Cf. next digression on "sites"][(1)] whose category structure comes from the preorder relation of divisibility.

The presheaves themselves would not have much topological significance so we need to introduce the exactness properties writing down the exactness conditions (3.14) in terms of commutative algebra:

Precisely:

For every $\forall f \in A$ and elements $f_i \in A$ $(i \in I)$ such that $X_f = \bigcup_{\in I} X_{f_i}$. I want that the diagram

(3.15)
$$G(f) \to \prod_{i\in I} G(f_i) \rightrightarrows \prod_{i,j} G(f_i f_j)$$

defined as before, in terms of the restriction morphisms, should be exact for every f and for every open covering, i.e. we should be able to identify $G(f)$ with the subset of the product $\prod_{i\in I} G(f_i)$ consisting of all the elements whose images by the two arrows are the same.

Now in order to interpret in terms of commutative algebra we have:

a) For every $X_{f_i} \subset X_f$ which means, I recall $\exists n_i \geq o, h_i \in A$ s.t. $f_i^{n_i} = fh_i$ i.e. $\bigcup_{i\in I} X_{f_i} \subset X_f \Rightarrow f_i^{n_i} = fh_i \quad i \in I$

(1) [I will formalize this oral exposition:
F: Op(X) → Sets was defined in the category of open sets of X.
G: $\mathbb{G}$ → Sets is defined in $\mathbb{G}$. However it makes sense to say that G is a contravariant functor $\mathbb{G}$ → Sets because the divisibility relation $g|f \Leftrightarrow \exists\, n \geq 0,\ h \in A$, s.t. $f^n = hg$ defines a categorical structure in A where Hom $(g,f) = \emptyset$ iff $g \nmid f$ and there is a unique morphism (preorder relation) iff $g|f$.]

b) We need to express the converse property: $X_f \subset \bigcup_{i \in I} X_{f_i}$. This means that $V(f) \supset \cap V(f_i)$ which means also $V(fA) \supset \cap V(f_iA)$. This implies an inclusion between the radical ideals $\sqrt{fA}$, $\sqrt{\Sigma f_iA}$:

$$V(\sqrt{fA}\,) \supset V(\sqrt{\sum_{i \in A} f_iA} \tag{3.16}$$

which means that there is an integer $n \geq 0$ and finitely many elements $g_i \in A$ such that

$$f^n = \Sigma\, g_i f_i \tag{3.17}$$

Now we have interpreted everything in terms of commutative algebra! In fact we have reconstructed the "_site_" of the open sets of type X_f in terms of commutative algebra.[1]

Another important property of the canonical bases needed later is their functoriality. We shall prove it in §4.

4. THE FUNCTOR Spec. In the previous §1,2,3 we assumed $X = \operatorname{Spec} A$ to be fixed. Let us check now that to any morphism

$$u: A \to B \tag{4.1}$$

in the category $\mathcal{C}$ of commutative rings with units corresponds a continuous map

$$\operatorname{Spec} u: \operatorname{Spec} B \to \operatorname{Spec} A \tag{4.2}$$

between the spectra (in reverse order). We know already how to define the

---

(1) Our friend forgot that not everybody followed his course in Topoi (which is not necessary to understand this course) but he added a very short explanation

"_A site is just a category plus covering data._"

A _Site_ is the same as a "GROTHENDIECK _topology_", i.e. a category where _fibered products exist_, in which the essential properties of coverings makes sense, just replacing intersections $U_i \cap U_j$ in X by $U_i \times_X U_j$. Cf. Ch. II, §1.2 for the formal definition of fibre products. For "GROTHENDIECK's topologies" the quickest short development is given in DIEUDONNE, Advances II, Page 407-11.

map between the underlying sets $\mathfrak{b} \mapsto u^{-1}(\mathfrak{b})$. We need to verify just the continuity of Spec u. Let $V(S)$ be a closed set of Spec A ($S \subset A$) . The condition: $(\mathrm{Spec}\ u)(\mathfrak{b}) = u^{-1}(\mathfrak{b}) \in V(S)$ is equivalent to $u^{-1}(\mathfrak{b}) \supset S \Rightarrow \mathfrak{b} = u(S) \Rightarrow \mathfrak{b} \in V(u(S))$, i.e. the inverse image of the closed set $V(S)$ of Spec A by Spec u is $V(u(S))$ which is closed in Spec B .

The induced map Spec u is a continuous map.

Spec is a contravariant functor from the category $\mathcal{C}$ of commutative rings with unit and unit preserving ring homomorphisms to the category of topological spaces and continuous maps.

As particular cases of (4.1) let us consider the surjective canonical homomorphism $A \xrightarrow{u} A/\mathfrak{a}$ ($\mathfrak{a}$ ideal of A) defined by $a \mapsto a + \mathfrak{a}$. Then Spec u: $\mathrm{Spec}(A/\mathfrak{a}) \to \mathrm{Spec}\ A$ is a canonical injection of the spectra.

If $\mathfrak{a} = \mathrm{Nil}\ A$ is the Nilradical of A , the bijection between the underlying sets Spec A and Spec(A/Nil A) defined by $\mathfrak{p} \mapsto \mathfrak{p}/\mathrm{Nil}\ A$ is a homeomorphism between both topological spaces:

(4.3) $$\mathrm{Spec}\ A \rightrightarrows \mathrm{Spec}(A/\mathrm{Nil}\,A)$$

This property alone makes clear that we cannot recover A from Spec A . The case that A is a field is even more expressive: the spectrum of any field k is a one-point space corresponding to the unique ideal (0) of k: Spec k = [(0)] .

Now we can prove that the continuous map (4.2) induced by (4.1) maps the basic open sets of Y (cf. §3) in those of X . In fact the inverse image $\varphi^{-1}(Y_f)$ of the open set Y_f ($f \in A$) is open in X . We shall prove the functoriality:

(4.4) $\varphi^{-1}(\mathcal{B}_Y) = \mathcal{B}_X$ $(\mathcal{B}_X, \mathcal{B}_Y$ canonical bases of $Op(X), Op(Y))$.

in the beautiful sharper form

$$\varphi^{-1}(Y_f) = X_{u(f)} \tag{4.5}$$

for every $Y_f \in \mathcal{B}_y$, $\forall f \in A$. Let us come back to the simplified notations of §3 used by GROTHENDIECK in his lectures, instead of the functorial ones used in this §4, i.e. we write again $X = \mathrm{Spec}\ B$, $Y = \mathrm{Spec}\ A$, $\varphi = \mathrm{Spec}\ u$ in such a way that Spec transforms $A \xrightarrow{u} B$ into $Y \leftarrow X$. Let $Y_f \in \mathcal{B}_Y$ be any basic open set of Y corresponding to any element $f \in A$. We have

$$\begin{aligned} Y_f = \{\mathfrak{p} \in \mathrm{Spec}\ A \mid f \notin \mathfrak{p}\} &= \varphi^{-1}(Y_f) = \{q \in \mathrm{Spec}\ B \mid \varphi(q) \in Y_f\} = \\ &= \{q \in \mathrm{Spec}\ B \mid u^{-1}(q) \in Y_f\} = \{q \in \mathrm{Spec}\ B \mid f \notin u^{-1}(q)\} = \\ &= \{q \in \mathrm{Spec}\ B \mid (f) \notin q\} = X_{u(f)} \ . \end{aligned} \tag{4.6}$$

q.e.d. In otherwords: The canonical basis $\mathcal{B}_y$ of Y is mapped naturally by φ^{-1} in the canonical basis $\mathcal{B}_X$ of X in such a way that (4.5) holds!

In order to recover A we shall define in Part II the structure sheaf $\tilde{A}$ on Spec A ; in such a way that $\Gamma(\mathrm{Spec}\ A, \tilde{A}) \cong A$. Then we shall see how these sheaves in the category $\mathcal{B}$ play a fundamental rôle.

5. DIGRESSION ON POINT SET TOPOLOGY APPLICABLE TO SPEC A . EXAMPLES.

There are several important, purely topological notions of old time algebraic geometry which are meaningful for arbitrary topological spaces. They are applied here to the highly non-HAUSDORFF spaces of algebraic geometry: algebraic varieties, schemas...etc., via the ZARISKI topology. They are not usually handled in courses on point-set topology because they become trivial in separated spaces, which are regarded as the "natural ones". The T_0, T_1 spaces needed here are neglected since they are "pathological"... The fact that they play so big a role in algebraic geometry makes it highly desirable to overcome this bias since the algebraic geometric objects are very natural and important objects of study. The typical cases treated below by GROTHENDIECK are irreducibility, generic point (already mentioned before), specialization, sober spaces, etc. For further details cf. EGA, Springer, Ch. 0, §2, page 48.

GROTHENDIECK's SUMMARY: I am going to prove now that the Spectrum $X = \mathrm{Spec}\ A$ is a sober space, i.e. for every closed irreducible(*) subspace Y of X there exists a point $y \in Y$ such that $Y = \overline{\{y\}}$, i.e. according to previous definitions and remarks every closed irreducible subspace has one and only one generic point. In other words, there exists a (1-1)-correspondence $x \to \overline{\{x\}}$ between the set of points of X and the family of closed irreducible subspaces of X.

A topological space X is called irreducible iff X is non empty and it cannot be decomposed as the union of two closed sets both different from X :

$$(5.1) \qquad X \neq \emptyset,\ X = X' \cup X'', \quad X' = \overline{X'}, \quad X'' = \overline{X''} \Rightarrow \begin{cases} X' = X \\ \text{or} \\ X'' = X \end{cases}$$

REMARK. Irreducibility is not quite the same as the connectedness property, because we did not assume $X' \cap X'' = \emptyset$! In other words, every irreducible space is connected but not conversely. Look for instance a the union of two intersecting lines $A \cup B$, $A \neq B$ in the affine plane. $A \cup B$ is connected but it is not irreducible.

A subspace Y of X is irreducible iff $Y \subset A \cup B$ (A and B closed in X) implies that Y is contained in A or $Y \subset B$. As a consequence $Y \subset X$ is irreducible iff its closure $\overline{Y}$ is irreducible; in other words irreducibility is stable under closure. A one-point set $\{x\} \subset X$ is irreducible for any X ; we saw already that $\{x\}$ is not necessarily closed in Spec A , in other words Spec A is not necessarily a T_1-space, i.e. we need to consider the possibility that $\overline{\{x\}}$ will be larger than $\{x\}$ (and often "much larger" cf. Examples at the end of §5). Because of the previous remark we have: The closure $\overline{\{x\}}$ of a one-point set is always irreducible. Conversely, let V be an irreducible subspace of X . Any point $x \in V$ such that $V = \overline{\{x\}}$ is called a generic point of V (Cf. §3 for the particular case of Spec A).

If $y \in \overline{\{x\}}$ every closed set F containing x contains also $y \Leftrightarrow$

(*) Cf. the topological definition of irreducibility given by (5.1). This definition is the same used by "old timers". The "new" fact is that this old definition can be can be interpreted as a general topological notion applied to the ZARISKI topology.

Every open neighborhood of y contains x. If X is a T_o-space (KOLMOGOROFF space) every irreducible subspace of V contains at most one generic point (because if $\overline{\{x\}} = \overline{\{y\}} = V$ and $x \neq y$ every open neighborhood of $x(y)$ contains $y(x)$ agains the T_o-property. (Cf. (*) of page (90)). We say that y is a specialization of x ($\Leftrightarrow$ x a generization of y) iff $y \in \overline{\{x\}}$.

A topological space is called sober iff every irreducible subspace Y of X has a generic point. If X is T_o and sober every irreducible subspace of X contains one and only one generic point. In other words:

In a T_o sober space X there is a bijective map $x \to \overline{\{x\}}$ between the points of X and the set of irreducible subsets of X. If $f: X \to Y$ is a continuous map and the source X is irreducible, then the image subset $f(X)$ on the target space is also irreducible.

For separated spaces irreducibility is not too interesting! A separated space X is irreducible iff X contains just one point!

In fact the set-theoretic dual of the definition property (5.1) is: Any two non-empty open sets of X intersect[1] (cf. I, §14):

$$A, B \text{ both open and } A \neq \emptyset,\ B \neq \emptyset \Rightarrow A \cap B \neq \emptyset$$

As a consequence an irreducible space containing two non empty open sets cannot be Hausdorff: if P, Q are two different points of Hausdorff space there exist two disjoint open neighborhoods of P, Q thus X cannot be irreducible.

In Algebraic Geometry, of course, the situation is entirely different! For instance for a classical affine algebraic variety X, X is irreducible iff the ring of polynomial on X is an integral domain.

The application of these notions to Spec A ($A \in \mathrm{Ob}\, \mathcal{C}$) endowed with its spectral topology (cf. §2) is immediate.

If X is the Spectrum of a ring $A \in \mathrm{Ob}\, \mathcal{C}$, we want to see what means

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(1) This property can be taken also as a definition of irreducibility. Other equivalent conditions are:
1) Every non-empty open set $A \subseteq X$ is dense in X: $\bar{A} = X$.
2) Every open subspace of X is connected. We leave the easy verification to the reader.

that X is irreducible. First let us remark that the spectrum of A doesn't change if we replace A by $A/\mathrm{Nil}A$, because every prime ideal contains $\mathrm{Nil}A$ and the map $\mathfrak{p} \mapsto \mathfrak{p}/\mathrm{Nil}A$ ($\forall \mathfrak{p} \in \mathrm{Spec}\, A$), (which is a bijection between $\mathrm{Spec}\, A$ and $\mathrm{Spec}\, A/\mathrm{Nil}A$), preserves the topology.

As a consequence, <u>in order to check irreducibility we can assume that A is a reduced ring</u>. Let us verify the two irreducibility conditions separately. That $X = \mathrm{Spec}\, A$ is non empty means $A \neq 0$, because $X = \phi$ means that A has no prime ideals and by KRULL's theorem $A = 0$. Now let us assume $X = X' \cup X''$ (X', X'' closed subspaces of X). X, X' and X'' corresponds to radical ideals $\mathcal{J}, \mathcal{J}'$ and the union corresponds to the product $\mathcal{J}'\mathcal{J}''$, i.e.

$$X' = V(\mathcal{J}'),\ X'' = V(\mathcal{J}'') \quad X = V(\mathcal{J}'\mathcal{J}'')$$

This latter property means $\mathcal{J}'\mathcal{J}'' = 0$, i.e. $\mathcal{J}'\mathcal{J}''$ is the zero ideal and there is a (1-1)-map between closed sets of X and radical ideals of A. Because of our assumption that A is reduced, the zero ideal is a radical ideal. Then the irreducibility property is equivalent to $\mathcal{J}'\mathcal{J}'' = 0$. This property is true iff A has no zero divisors. In other words (for a reduced A) <u>$X = \mathrm{Spec}\, A$ is irreducible iff A is an integral domain</u>. In general, we have:

X <u>irreducible</u> $\Leftrightarrow$ $A/\mathrm{Nil}\, A$ <u>is an integral domain</u>.

We know that any closed subspace of $\mathrm{Spec}\, A$ can be identified with $\mathrm{Spec}\, A/\mathfrak{a}$ where $\mathfrak{a}$ is a radical ideal of A. We have: the closed subspace $V = \mathrm{Spec}\, A/\mathfrak{a}$ of $\mathrm{Spec}\, A$ is irreducible iff $\mathfrak{a}$ is prime. Then <u>$\mathfrak{a}$ is a generic point of $\mathrm{Spec}\, A$</u>.

Every irreducible closed set of Spec A has a unique generic point
The existence was just proved! The uniqueness is a consequence of the T_o-property of Spec A (if an irreducible Y of a T_o sober space conta two different generic points y, y' every open neighborhood of one of t should contain the other, in contradiction with T_o !

The topological space Spec A is T_o and sober.
We already saw the proof: if every open U containing x also contai y ($\neq$ x) then every closed containing x also contains y $\Leftrightarrow$ $\mathfrak{p}_y \supset \mathfrak{p}_x$ Then cannot $\mathfrak{p}_x \supset \mathfrak{p}_y$ otherwise be x = y . Thus Spec A is T_o . On other hand Spec A/$\mathfrak{a}$ ($\mathfrak{a}$ radical ideal of A) is irreducible if $\mathfrak{a} = \mathfrak{p}$ is prime and we already saw that $\mathfrak{p}$ is generic for V and it is the o one because Spec A is T_o .

Any irreducible subspace I of a topological space is contained i maximal one (which is necessarily closed) because the family of irreduc subspaces $\supset$ I is partially ordered by $\supset$ and it is inductive (= eve totally ordered subset is bounded). Then as a consequence of ZORN's lem there exist maximal elements. As a consequence we have:

Any topological space X can be decomposed as an irredundant[1] u of irreducible closed subspaces:

$$(5.2) \qquad X = \bigcup_{\nu \in J} I_\nu \qquad (I_\nu \text{ irreducible, } \forall \nu \in J)$$

This decomposition in irreducible components I_ν is unique (up to the ordering).

In classical Algebraic Geometry the set of irreducible components w finite because the rings A were Noetherian (cf. Ch. II,). More

(1) Irredundant means: every I_ν is not contained in $\bigcup_{\mu \neq \nu} I_\mu$.

generally we can prove the finiteness of (5.2) if X is Noetherian (cf. def. below):

A topological space X is called <u>Noetherian</u> iff anyone of these equivalent conditions hold:

1) Descending chain condition for closed subspaces (in its two equiforms):

$X_1 \supset X_2 \supset \dots$ is stationary $\qquad (X_i = \overline{X}_i \ \forall i)$

$X_1 \supsetneqq X_2 \supsetneqq \dots$ is finite

1)' Ascending chain condition pen sets U_i

$U_1 \subset U_2 \subset \dots$ stationary $\qquad U_i$ open for every i

$U_1 \subsetneqq U_2 \subsetneqq \dots$ finite $\qquad U_i$ open for every i

<u>If A is Noetherian Spec A is Noetherian also.</u> GROTHENDIECK shows in EGA that the converse property is not true.

Accordingly there are finitely many minimal ideals of a Noetherian ring A. They are the generic points of the irreducible components of Spec A.

The proof of the finiteness of the decomposition in irreducible components can be done by "Noetherian induction" as follows:

Let X be a Noetherian topological space. If X is irreducible there is nothing to prove. If $X = X' \cup X''$ and the finiteness property is false for X it should be false also for at least one of the closed subsets X', X'' of a non trivial decomposition $X = X' \cup X''$. Then $X' = X' \cup X_2'$ with X_1', X_2' closed and $\neq X'$. Let us assume that the is false for $X_1', \dots$ Then we construct an infinite strictly descending chain of closed sets: $X \supset X' \supset X_1' \supset \dots$ against the "Noetherianity".

EXAMPLES. I want to present two kinds of examples:

1° Those coming from a classical affine algebraic variety V (not necessarily irreducible) over the complex number field. V will appear as the subspace of closed points of $\operatorname{Spec} A(V)$ where $A(V)$ is the ring of $\mathbb{C}$-valued polynomial functions $f: V \to \mathbb{C}$ defined on V.

2° Examples which have no classical analogon.

It is clear that examples of the first type can be particularly useful for the readers with some background in Classical (or "old time") Algebraic geometry. But since I worry about "applicability", I cannot believe that a non expert in classical Algebraic Geometry is ignorant of lines, planes, algebraic curves, surfaces in $\mathbb{C}^3$. I cannot believe that the reader never heard of the conic sections (is it fair to assume that the reader never heard of KEPLER laws in this age of space exploration?).... However, restricting ourselves exclusively to spectra coming from classical varieties would not be in the best interest of the reader. Such a restriction could lead to the false idea that all Spectra come from algebraic varieties...

Since the topological spaces $\operatorname{Spec} A$ and $\operatorname{Spec} A/\operatorname{Nil} A$ are homeomorphic we can assume that all the commutative rings with units mentioned in the rest of this section are reduced ($\Leftrightarrow$ $\operatorname{Nil} A = 0$).

1° SPECTRA COMING FROM COMPLEX AFFINE ALGEBRAIC VARIETIES.

A reduced ring A of type $A = \mathbb{C}[\xi_1, \xi_2, \ldots, \xi_n]$, i.e. a finitely generated $\mathbb{C}$-algebra without nilpotent elements ($\neq 0$), represents an affine algebraic variety V. Since A is Noetherian V can be decomposed as an irredundant union of finitely many irreducible components $I_1, I_2, \ldots, I_h$

$$V = I_1 \cup I_2 \cup \ldots \cup I_h$$

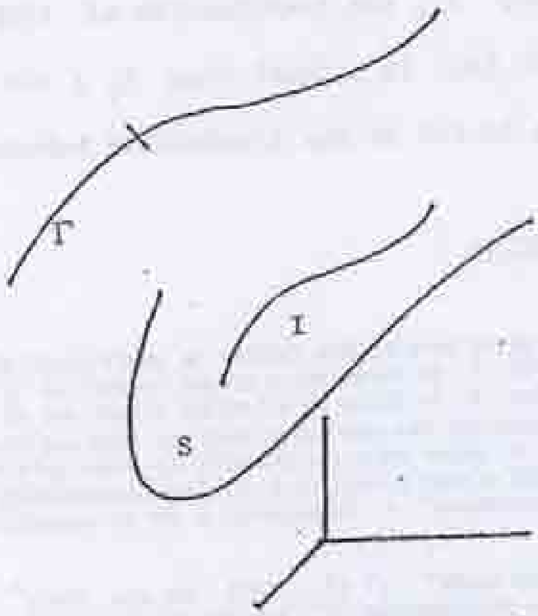
The points of V are the maximal ideals of A: $V = \text{Max } A$. The irreducible components I_ν $(\nu = 1,2,\ldots,h)$ correspond bijectively to the (finitely many!) minimal ideals of A: $\mu_1, \mu_2, \ldots, \mu_h$ and I_ν can be identified with the maximal spectrum $\text{Max } A/\mu_\nu$ $(\nu = 1,2,\ldots,h)$. A/μ_ν is an integral domain. Its field of quotients has a finite transcendence degree over $\mathbb{C}$, equal by definition to the complex dimension of I_ν. A choice of a finite set of generators for A, for instance ξ_j $(j = 1,2,\ldots,n)$ enables us to define a surjective homomorphism $\mathbb{C}[T_1,T_2,\ldots,T_n] \to A$ $T_j \to \xi_j$, $j = 1,\ldots,h)$ which provides an affine embedding $V \to \mathbb{C}^n \ldots$, but let us stress once again that the embedding itself is not important. For a given A n might be variable $(n > \text{Max dim } A/\mu_\nu$ $\nu = 1,2,\ldots,h)$.

The enlargement $\text{Max } A \to \text{Spec } A$ might add to V "many more" "thicker" points: More precisely, one for every irreducible subvariety I of V (with $p_I \neq p_J$ if $I \neq J$. I.e. there is one point $p_\nu = p_{I_\nu}$ $\nu = 1,2,\ldots,h$ for each irreducible component. Every $x \in \text{Spec } A$ is generic for the subscheme A/p_x consisting of all the points of $\text{Spec } A$ representing irreducible subvarieties of the irreducible variety V_x represented by x, i.e. $y \in \overline{[x]}$ is equivalent to $V_y \hookrightarrow V_x$. In particular for the μ_ν, $p_\nu \in \text{Spec } A$ is the generic point of $\overline{[p_\nu]}$ whose points represent all the irreducible subvarieties of V contained in the ν-th irreducible component I_ν $(\nu = 1,2,\ldots,h)$.

In classical algebraic geometry irreducible varieties played a privilege rôle... Let us examine in particular the cases of an irreducible affine curve Γ, an irreducible affine surface S and the affine space $\mathbb{C}^3$. Let $A(\Gamma), A(S), A(\mathbb{C}^3)$ be the corresponding $\mathbb{C}$-algebras of $\mathbb{C}$-valued polynomial functions. We have:

a) Spec $A(\Gamma)$ contains Γ plus the generic point (= zero ideal of $A(\Gamma)$).

(0) $\in$ Spec $A(\Gamma)$ because $A(\Gamma)$ is an integral domain.



b) Spec $A(S)$ contains S (= set of all the maximal ideals of $A(S)$) plus all the points representing irreducible curves lying on S, plus the generic point of S (= (0) $\in$ Spec $A(S)$).

c) Spec $A(\mathbb{C}^3)$ contains (besides the closed points of 3-space $\mathbb{C}^3$) the points representing all the irreducible curves, those representing all the irreducible surfaces and the generic point (0) $\in$ Spec $A(\mathbb{C}^3)$.

Forgetting about $\mathbb{C}$, if k is any commutative field, Spec k contains just one point (which is generic for Spec k). Thus once again there is no way of recovering k from Spec k...!

2° SPEC $\mathbb{Z}$ AND Spec $\mathcal{O}$ ($\mathcal{O}$ LOCAL RING):

Since $\mathbb{Z}$ is the initial object of $\mathcal{C}$ let us consider Spec $\mathbb{Z}$! We have (0) $\in$ Spec $\mathbb{Z}$ ((0) is prime $\Leftrightarrow$ $\mathbb{Z}$ is an integral domain). (0) is the generic point of Spec $\mathbb{Z}$. Any other point $\neq$ (0) of Spec $\mathbb{Z}$ has the form (p) where p is any prime number: $p = 2,3,5,7,10,\ldots$. Any such (p) is a maximal ideal, i.e. Max $\mathbb{Z} = \{(p) \mid p \text{ prime}\}$. Thus every (p) is a closed point.

Our last example will be the spectrum, Spec A, of a local ring with maximal ideal $\mathfrak{m}$. These rings were considered in classical algebraic geometry even though they are not of the type $\mathbb{C}[\xi_1,\xi_2,\ldots,\xi_n]$. Thus, Spec A is extremely different from Spec $\mathbb{C}[\xi_1,\dots,\xi_n]$ (except in the trivial case). In fact: Spec A contains just one closed point $\mathfrak{m}$. Any other point $\mathfrak{p} \in$ Spec A is a generalization of $\mathfrak{m}$ ($\Leftrightarrow \mathfrak{m} \in \overline{[\mathfrak{p}]}$). In classical algebraic geometry if Y is an irreducible subvariety of the complex irreducible algebraic variety X, the localization of $A(X)$ with respect to the prime ideal $\mathfrak{p}_Y$ (cf. §8) is a local ring A_Y; the points of Spec A_Y correspond bijectively to all of the irreducible subvarieties Z; $Y \subset Z \subset X$.

HISTORICAL EVOLUTION OF GENERICITY

During the Italian period a property P of an irreducible (affine or projective) algebraic
variety X was "true for the generic point" iff P is false for a proper subvariety $Y \subset X$.
In today language this is equivalent to say that P is false in a ZARISKI closed set of X
($\Leftrightarrow$ P is true in a ZARISKI open set). For instance the statement: there is just one line containing
a generic ordered pair of points in the plane X means that the property is false only for the
(x,y) diagonal $\{x \times x \mid x \in X\}$ which is a ZARISKI closed subset of $X \times X$. This conventional language
is parallel to the properties "valid almost everywhere" = (except for a set of measure zero...)
in real analysis... .

VAN DER WAERDEN introduced his "allgemeine Punkt" $(\xi^1,\xi^2,\dots,\xi^n)$ for any $X \subset k^n$ (k ground
field), as a "set of generators of the k-algebra representing X", but the characteristic property
is $(\xi) \in \Omega$ (any extension field of k) is a generic point of X iff [1] $f(\xi) = 0$ for any f
vanishing identically in X [2] if x is any other point of X $f(\xi) = 0 \Rightarrow f(x) = 0$. The purpose
of this allgemeine Punkt is to try to make "algebraic" the previous Italian notion (where an "actual"
generic point was not really defined) but it is clear that ξ uses the coordinates; it is not unique...
GROTHENDIECK's genericity assigns a unique generic point to every irreducible subset of Spec A and
this point is "intrinsic" (no affine embeddings are needed), x (generic point of X) defined in the
text by $X = \overline{\{x\}}$ is then a purely topological notion. Because of the definition of the ZARISKI
topology it is still true that $f(x) = 0 \Rightarrow f(y) = 0$ for any specialization $y (\in \overline{\{x\}})$, as in
VAN DER WAERDEN.

PART II

SHEAVES ON AFFINE SCHEMES

6. INTRODUCTION.

The main properties of sheaves of abelian groups, rings, etc. on the category of topological spaces are supposed to be known (cf. PREREQUISITES, page 6). I believe that the standard Bibles: GODEMENT, the complements in EGA are too extensive for a beginner. They are reference books. The main reasonable short introduction with full proofs is SERRE's FAC paper; (HIRZEBRUCH's sketchy introduction, or the extended one in GUNNING-ROSSI, ... are also reasonably "short"...). A short introduction on sheaves of sets is provided in the Appendix (§14).

The main result of this Part II of Ch. III is the fact that the structure sheaf defined on Spec A in terms of the localizations $A_{\mathfrak{p}}$ ($\mathfrak{p} \in \operatorname{Spec} A$) transforms Spec A into a locally ringed space (cf. I, §0, page 5) (Spec A, $\tilde{A}$) and A becomes the ring of global sections of $\tilde{A}$, thus we recover A, from the affine scheme attached to A.

The category Aff of affine schemes is a full subcategory of the category of locally ringed spaces. Aff is isomorphic with the opposite category $\mathcal{C}^{\circ}$ of the category $\mathcal{C}$ of commutative rings with unit.

The topological structure on X (= on Σ) is only a small part of the structure needed on X, in order to recover A. In fact, X is a ringed space[1]. We are going to construct a covariant functor: Aff $\to$ $\mathcal{L}rs$ which goes from the category Aff of affine algebraic spaces to the category of locally ringed spaces,[2] or equivalently a contravariant functor from the opposite category $\mathcal{C}$ of commutative rings with units to $\mathcal{L}rs$.

Now I will define this extra structure that I promised, namely I will define a sheaf of rings $\mathcal{O}_X = \tilde{A}$ on the spectrum of A, called the structure sheaf of X.

A reader with the elementary sophistication of sheaf theory given in FAC can readily understand the construction as follows:

The stalk $\tilde{A}_{\mathfrak{p}}$ of $\tilde{A}$ at a point $\mathfrak{p} \in \operatorname{Spec} A$ is the direct limit

$$\tilde{A}_{\mathfrak{p}} = \varinjlim_{U \in \operatorname{Ob} \mathcal{B}} A_U \tag{6.1}$$

of the localized rings[3] A_U of A where U runs in the category $\mathcal{B}$ of basic open sets of Spec A (cf. Ch. III, §3)[4]. (A_U is the largest homomorphic image of A ($S_U: A \to A_U$) characterized by the universal property that if $U = X_f$ $S_U(f)$ is invertible in A_U).

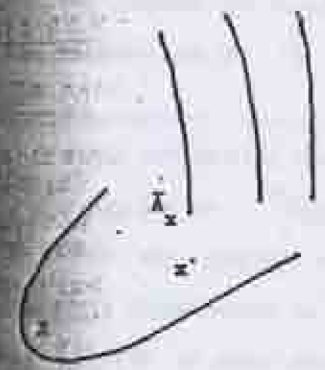
(1) Cf. def. in footnote (8), page 0-4 of the Introduction. Cf. EGA-Springer, Ch. 0, §4 Espaces annelés, page 37. We shall define the morphisms in §7.

(2) An object of $\mathcal{L}rs$ is a ringed space $(X, \mathcal{O}_X)$, whose structure sheaf $\mathcal{O}_X$ is a sheaf of local rings, i.e. for every $x \in X$ the stalk at x, $\mathcal{O}_{X,x}$ is a local ring. Morphisms in $\mathcal{L}rs$ will be defined accordingly in §7.

(3) It is clear that the reader should be aware of the theory of localization (which is standard equipment in many graduate algebra courses, cf. for instance LANG or ATIYAH-MCDONALD and BOURBAKI-COMM-ALG, for full details). The main definitions were reviewed by GROTHENDIECK when needed. I have collected them in §8.

(4) Cf. §3.

The rings of continuous sections of $\tilde{A}$ over a basic open set U is isomorphic with A_U. If $U = X_f$, A_U is isomorphic with A_f (1). In other words any fraction a/s of A_U can be written in the form a/f^n for any generator of U. In particular, for $f = 1$, $X_1 = X \Rightarrow A_1 = A$, i.e. we recover A as the ring of global continuous sections of the structure sheaf $\tilde{A}$ over X.



REMARK. In the FAC approach sheaves can be defined inde-pendently of presheaves, by using stalks from the very beginning. This justifies this agricultural terminology for the "fibers" $p^{-1}x$ in all languages (faisceaux, Garben, etc) GROTHENDIECK defines a sheaf as a particular presheaf satisfying certain exactness conditions. The reason for this is technical. In his "sites" (= GROTHENDIECK topologies) there are not necessarily points of an object X, as a consequence there are not necessarily "fibers" (stalks $p^{-1}(x)$). In §7 I will write a short summary establishing the equivalence of both points of view. In §8 I review the essential facts concerning rings of fractions needed to follow the details of the construction of $\tilde{A}$ and sheaves of $\tilde{A}$-modules attached to the A-module, M.

Let M be any A module. Then we can construct a sheaf $\tilde{M}$ of $\tilde{A}$-modules whose stalk at every $\mathfrak{p} \in \operatorname{Spec} A$ is the localized module $M_{\mathfrak{p}}$.

Actually both constructions of $\tilde{A}$ and $\tilde{M}$ are done simultaneously by using the fact that the localization theory for A can become a particular case (when A is regarded as an A-module over itself). Conversely, if $\tilde{M}$ is any sheaf of $\tilde{A}$ modules (i.e. every $\tilde{M}_x$ is an $\tilde{A}_x$-module for $\forall x \in \operatorname{Spec} A$) the natural question arises whether or not $\tilde{M}$ comes from an A-module. This property holds iff $\tilde{M}$ is a quasi-coherent sheaf of $\tilde{A}$-modules.

The actual construction of $\tilde{A}$, inspired by the tape recording is given in §9. Preparatory material is developed in §7 and §8. A knowledgeable reader should start with §9.

Then we shall be able to define finally the category $\tilde{G}^*$ of affine schemes, which is, in fact, isomorphic to the category *Aff* of algebraic spaces (cf. Ch. I) (i.e. the category of representable functors on G), or what is the same we can identify the affine schemes with the objects of the category opposite to G.

In order to make this explicit we shall attach to every commutative ring with unit $A(\in \operatorname{ob} G)$ the affine scheme $(\operatorname{Spec} A, \tilde{A})$, i.e. the locally ringed space whose underlying topological space is $\operatorname{Spec} A = \{\mathfrak{p} \mid \mathfrak{p} \text{ prime ideal of } A\}$ endowed with the spectral topology and whose structure sheaf $\tilde{A}$ is the sheaf of local rings $A_{\mathfrak{p}}$ ($\mathfrak{p} \in \operatorname{Spec} A$) just mentioned. We shall

(1) Remember that $X_{fg} = X_f \cap X_g$, i.e. $\mathcal{B}$ is stable by finite intersections. In order to define the "germs" (6.1) it is sufficient to consider $\mathcal{B}$ rather than the whole family $Op(X)$, because $\mathcal{B}$ is cofinal.

$A_f = S_f A$ denotes the ring of fractions a/f^n of A with denominators in the multiplicative set $S_f = \{f^n \mid n \geq 0\}$.

see that $(\mathrm{Spec}\ A, \tilde{A})$ gives back A[4]; moreover, if $A \xrightarrow{f} B$ is any morphism in $\mathcal{G}$ there is a well defined morphism of locally ringed spaces in the opposite direction: $(\mathrm{Spec}\ B, \tilde{B}) \to (\mathrm{Spec}\ A, \tilde{A})$ f* defines a continuous map $\mathrm{Spec}\ B \to \mathrm{Spec}\ A$ between the underlying topological spaces and a sheaf homomorphism (reversing the direction again).

It is important to remark the fact that A might or might not be reduced.

7. GENERALITIES ON SHEAVES FOLLOWING FAC AND GROTHENDIECK, RINGED SPACES

Since I believe that FAC is reasonably well-known, as a kind of transition from classical to preparation for GROTHENDIECK's algebraic geometry and the definition of a sheaf is given there directly without introducing first presheaves, we shall see the equivalence of both approaches.

Following FAC a sheaf of abelian groups [1] on a topological space X is an <u>étale covering</u>[2] $p: S \to X$ of another topological space S <u>onto</u> X, such that for every $x \in X$ the <u>stalk</u> $S_x = p^{-1}(x)$ is an abelian group an the map of the fibre product $S \times_X S \to X$[3] (cf. Ch. II, §3) defined by

$$(a,b) \mapsto a-b \tag{7.1}$$

is continuous.

There are two ways of comparing sheaves and presheaves in FAC: If S is a sheaf, the map $U \mapsto \Gamma(S|U)$ assigning to every open set of X the abelian group of continuous sections [4] is a presheaf, (Cf. §3) calle canonical presheaf defined by S or the presheaf of local continuous sections. On the other hand if P is an arbitrary presheaf of abelian groups we can define the germs at x $(\in X)$ of an element $f \in P(U)$ (U op

---

[4] For instance the trivial case that $A = k$ is a field $(\mathrm{Spec}\ k, \tilde{k}) = (\{0\}, k)$!

[1] Similarly we can define sheaves of rings, local rings, sets, etc.

[2] In this context <u>étale covering</u>, means just that the projection $S \xrightarrow{p} X$ is a local homeomorphism, i.e. for every $s \in S$ there exists an open neighborhood $U \ni s$ such that $p|U: U \to p(U)$ is a homeomorphism. Cf. Appendix for further information.

[3] In the FAC definition of sheaves $S \times S$ has the product topology and $S \times_X S$ has the topolo induced by $S \times S$.

[4] A continuous local section of the sheaf S on U (open in X) is a continuous map $s: U \to$ such that $p \circ s = 1_U$.

$\ni x)$ by a well-known equivalence relation that can be defined as a direct limit $P_x = \varinjlim_{x\in U} P(U)$. Then we construct the limit sheaf S_P of the presheaf P on X by means of the disjoint union $S_P = \bigcup_{x\in X} P_x$ or underlying set, by introducing a natural topology on S_P such that if $x \in X$ becomes an étale covering of X and the algebraic operations become continuous. These two functors between the categories Presh (X) and Top(X) of sheaves and presheaves are not inverse of each other. Precisely we are concerned with the fact that the presheaf $\mathcal{P}$ of continuous local sections of the limit set S_P of a presheaf P cannot be identified canonically with P , but there is just a canonical map $P \to \mathcal{P}$. The natural problem arises of characterizing the presheaves P which can be identified with the canonical presheaf of S_P . This problem was already stated and solved in FAC. The solution can be characterized by an exactness condition:

The presheaf P is canonical iff for any open covering $U = \bigcup_{i\in I} U_i$ the following diagram is exact:

$$F(U) \to \prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j) \tag{7.1}$$

We recommend expressing in words the two verifications involved in this exactness condition, checking that we get back the old characterization of canonical sheaves by the two conditions:

a) F is separated $\Leftrightarrow$ $F(U)$ is uniquely determined by all its restrictions.

b) F is "local" i.e. for every map $U_i \mapsto F(U_i)$ satisfying the "matching condition" there exists a section (unique because of a) such that $F(U_i) = F(U) \mid U_i$.

The two functors

$$\mathrm{Top}(X) \underset{\gamma}{\overset{i}{\rightleftarrows}} \mathrm{Presh}(X) \qquad (i \text{ inclusion, } \gamma(P) \text{ canonical sheaf})$$

between the two categories[(1)] Top(X) and Presh(X) of sheaves and presheaves over X are adjoint to each other: (cf. Ch. II), i.e. if S is a sheaf and P a presheaf we have canonical identifications:

$$(7.2) \qquad \underset{\mathrm{Top}(X)}{\mathrm{Hom}}(S,\gamma(P)) \simeq \underset{\mathrm{Presh}(X)}{\mathrm{Hom}}(i(S),P)$$

In this course (and in most of his publications) GROTHENDIECK takes the exactness condition as definition of a sheaf, because it is more categorical, i.e. <u>we can replace Op(X) by any category with fiber products</u>; summarizing: <u>a sheaf over X is a presheaf satisfying the exactness condition</u>.

With this approach the "stalks" do not appear explicitly but this is convenient for arbitrary "<u>sites</u>" (= GROTHENDIECK's Topologies) where the stalks do not play any primary role.

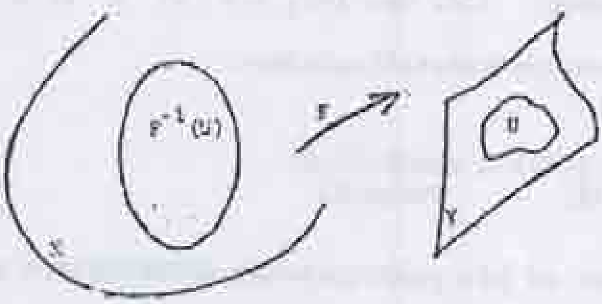
The main objects of study of GROTHENDIECK's algebraic geometry, the <u>schemes</u>, are particular instances of <u>ringed spaces</u>, i.e. of topological spaces X with a structure sheaf of rings $\mathcal{O}_X$; then they will be denoted by a pair $(X,\mathcal{O}_X)$. The simplest examples of such structures known to the reader, at least informally,[(2)] are manifolds of various types (topological or C^∞-differentiable in the sequel, to fix the ideas). Let us think of a surface X , for instance. The local real valued functions on X form a presheaf $U \mapsto \Gamma(U|\mathcal{O}_X)$ (U open in X , $\Gamma(U|\mathcal{O}_X)$ is the ring of functions $U \to \mathbb{R}$).

---

(1) The morphisms in Presh(X) are natural transformations. A morphism between sheaves. $S \xrightarrow{p} X$, $T \xrightarrow{q} X$ (defined as in PAC) is a continuous map $S \to T$ making commutative the triangle S,T,X and inducing fiber homomorphisms $S_x \to T_x$ for every $x \in X$.

(2) We do not assume any specialized knowledge on manifolds. It is enough to recall examples from "Freshman Calculus" to replace curves or surfaces, by manifolds of dimension n.

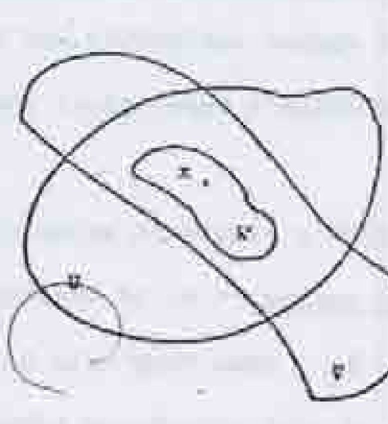
A map $(X,\mathcal{O}_X) \xrightarrow{F} (Y,\mathcal{O}_Y)$ between two surfaces means a continuous map $X \to Y$ which transforms local functions in local functions, and since functions behave in a contravariant way the only sensible way of doing so is to lift a local function $\varphi: U \to \mathbb{R}$ (U open in Y) to a local function



$\varphi \circ F : F^{-1}(U) \to \mathbb{R}$ (where $F^{-1}(U)$ is open in, because F is continuous). More precisely we have a ring homomorphism:

$$\Gamma(U,\mathcal{O}_Y) \to \Gamma(F^{-1}U,\mathcal{O}_X)$$

Moreover, this operation is compatible with the construction of germs (elements of $\lim\limits_{\overrightarrow{U \ni x}} \Gamma(U)$). Let us make explicit the construction:



Let $f_U: U \to \mathbb{R}$, $f_V \to \mathbb{R}$ be two local functions[1] defined in two open neighborhoods of x. We write $f_U \sim f_V$ iff there exists a third open set $x \in W \subset U \cap V$ such that $f_U|W = f_V|W$. It is easy to check that ~ is indeed an equivalence relation (in the set of local functions defined in open neighborhoods of x) compatible with the ring operations. The equivalence class defined by f_U is called the germ of f_U at x. Germs at x will be denoted by capital script letters $\mathcal{F}, \mathcal{G}, \mathcal{H}, \ldots$ with the left subscript x: ${}_x\mathcal{F}, {}_x\mathcal{G}, {}_x\mathcal{H}, \ldots$ This should not be confused with the real number ${}_x\mathcal{F}(x)$ (value of ${}_x\mathcal{F}$ at x) which makes sense since $f_U \sim f_V \Rightarrow f_U(x) = f_V(x)$.[2]

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[1] Continuous ($= C^0$) or C^∞ according to the case. The C^0 case is automatic. The C^∞ implies a strong restriction. Not every φ should be lifted, just those which are C^∞.

[2] But we cannot define ${}_x\mathcal{F}(y)$ for $y \neq x$.

The set of germs of functions at x has a natural ring structure. This ri
$\mathcal{O}_{X,x}$ of germs at x contains $\mathbb{R}$ as a subfield when we identify any $r \in$ I
with the germ $\mathfrak{F}$ of a locally constant function equal to r ($r \neq s \Rightarrow {}_r\mathfrak{F} \neq$

The map $f_U \to {}_x\mathfrak{F}$ is a homomorphism of commutative rings with unit. Moreover, the map ${}_x\mathfrak{F} \to {}_x\mathfrak{F}(x)$ is also a real valued ring homomorphism, whos kernel m_x is a maximal ideal of $\mathcal{O}_{X,x}$. In fact if $f_U(x) \neq 0$ it remains $\neq 0$ in a certain open neighborhood of x, as a consequence the corresponding germ $\mathfrak{F}_x$ is invertible, i.e. $\mathfrak{F}_x^{-1} \in \mathcal{O}_{X,x}$, in other words: the set $\mathcal{O}^*_{X,x} = \{\mathfrak{F}_x \in \mathcal{O}_{X,x} | \mathfrak{F}_x(x) \neq 0\}$ consists of all the invertible elements of

$\mathcal{O}_{X,x}$. The complementary set $m_x = \{\mathfrak{F}_x \in \mathcal{O}_{X,x} | \mathfrak{F}_x(x) = 0\}$ is a maximal ideal m_x. Commutative rings with unit having such a property are called <u>local rings</u> precisely because they appear naturally in the local study of manifolds in a sufficiently small neighborhood of any fixed point $x \in X$

It is clear that for any couple of corresponding points x, $F(x)$ $x \in X$, $F(x) \in Y$ we can lift any germ $\mathfrak{F}_{F(U)}$ at $F(x)$ to a germ at x: $\mathfrak{F}_{F(x)} \to \mathfrak{F}_{F(x)} \circ F$ <u>preserving the values</u>; in other words we define a map between the two local rings $\mathcal{O}_{Y,F(x)} \to \mathcal{O}_{X,x}$ making the diagram below

(7.3)
$$\begin{tikzcd}
\mathcal{O}_{Y,F(x)} \arrow[r, "F^*"] \arrow[d] & \mathcal{O}_{X,x} \arrow[d] \\
\mathbb{R} \arrow[r, "\mathrm{id}_{\mathbb{R}}"'] & \mathbb{R}
\end{tikzcd}$$

(1) In §8 we present a short review of a purely algebraic approach to local rings following the tape. In the meantime this short summary by GROTHENDIECK can help :...I recall that a commutative ring with identity is a local ring iff there is just one maximal ideal, or one can also say that the non-invertible elements are stable under sum and <u>a fortiori</u> they form an ideal (because in any ring A, if $a \in A$ is non-invertible, λa is still non-invertible for any $\lambda \in A$) and this ideal would be <u>the maximal ideal</u>. This can also be phrased as follows: For any two elements $f, g \in A$, $f + g$ invertible implies that either f or g is invertible, and multiplying this by the inverse of this sum amounts to saying that either f or $1 - f$ must be invertible for every f. Replacing f by $-f$: either f or $1 + f$ must be invertible.

commutative, where the vertical arrows map germs in the corresponding value at $F(x)$, x. In particular, if a germ vanishes at $F(x)$ its image vanishes at x also. In other words: <u>The image of the maximal ideal of</u> $\mathcal{O}_{Y,F(x)}$ by F^* <u>is contained in the maximal ideal m of</u> $\mathcal{O}_{X,x}$; or what is equivalent: the inverse image $F^{*-1}(m)$ of the maximal ideal of $\mathcal{O}_{X,x}$ is the maximal ideal of $\mathcal{O}_{Y,F(x)}$.[1]

These considerations suggest the right definition of <u>morphism of ringed spaces</u>:

Let $(X,\mathcal{O}_X)$, $(Y,\mathcal{O}_Y)$ be two <u>ringed spaces</u>.[2] A morphism $(X,\mathcal{O}_X) \xrightarrow{G} (Y,\mathcal{O}_Y)$ is a pair $G = (f, \theta)$ where $f: X \to Y$ is a continuous map between the underlying topological spaces and θ is an f-homomorphism $\theta: \mathcal{O}_Y \to \mathcal{O}_X$ between the sheaves, i.e. for every $x \in X$, there is a ring homomorphism $\theta: \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$.

It is clear how we can construct the <u>category</u> $\mathcal{R}s$ <u>of ringed spaces</u> with the $(X,\mathcal{O}_X)$ as objects and the morphisms are the previously defined.

The former example shows how we should sharpen the definition for <u>locally ringed spaces</u>. A ringed space $(X,\mathcal{O}_X)$ is called <u>locally ringed.</u> iff the stalks $\mathcal{O}_{X,x}$ at any x is a local ring.

---

(1) This situation suggests the algebraic definition of local homomorphism between two local rings $\mathcal{O},\mathcal{O}'$. If m, m' are the corresponding maximal ideals of $\mathcal{O},\mathcal{O}'$ and $h: \mathcal{O} \to \mathcal{O}'$ is an arbitrary homomorphism in $\mathcal{C}$, $h^{-1}(m') \subset m$, but the prime ideal $h^{-1}(m') \in \operatorname{Spec} \mathcal{O}$ might be $\neq m$. h is <u>local</u> iff $h^{-1}(m') = m$. This property is equivalent to the fact that the direct image $h(m)$ should be contained in m'. We shall come back to this following the tape.

(2) i.e. X,Y topological spaces $\mathcal{O}_X,\mathcal{O}_Y$ sheaves of rings (non necessarily local).

The locally ringed spaces from a subcategory $\mathfrak{Lrs}$ (not full!) of the category $\mathfrak{Rs}$ such that if $(X,\mathcal{O}_X)$, $(Y,\mathcal{O}_Y)$ belong to $\mathrm{Ob}(\mathfrak{Lrs})$ a morphism $(X,\mathcal{O}_X) \xrightarrow{G} (Y,\mathcal{O}_Y)$ in $\mathfrak{Rs}$ belongs to $\mathfrak{Lrs}$ iff $G = (f, \theta)$ is <u>local</u>, i.e. iff for every $x \in X$, the induced morphism $\mathcal{O}_{Y,f(x)} \xrightarrow{\theta} \mathcal{O}_{X,x}$ between these two local rings is <u>local</u>, i.e. it satisfies either one of the two equivalent conditions treated below: (cf. footnote (2)

1) The direct image of $m_{f(x)}$ is contained in m_x :

$$\theta_x(m_{f(x)}) \hookrightarrow m_x \tag{7.4}$$

$$\theta_x^{-1}(m_x) = m_{f(x)} \tag{7.5}$$

HISTORICAL REMARK.

The locally ringed spaces appear very naturally in the study of manifolds of various kinds, (cf. I, §0) for instance for complex analytic manifold $(X, \mathcal{O}_X)$ is the ring of germs of local holomorphic functions. SERRE's FAC paper introduced them in Classical Algebraic Geometry over an algebraically closed field. Then an affine algebraic variety in FAC's sense $(X, \mathcal{O}_X)$ is the following particular case:

X is the maximal spectrum of a finitely generated k-algebra A without nilpotent elements

$$X = \mathrm{Max}\, A \qquad A = k[\xi_1, \xi_2, \dots, \xi_n] \qquad \mathrm{Nil}\, A = 0$$

In particular X is irreducible iff A is an integral domain; $\mathcal{O}_X$ is the sheaf of germs of local regular functions on X (i.e. restrictions of the old "rational" or meromorphic functions to ZARISKI open sets where they have no poles and no "indeterminacy points," i.e. where they are actually defined.

8. DIGRESSION ABOUT RINGS AND MODULES OF FRACTIONS, LOCAL RINGS, LOCALIZATIONS.

⌈I present some preliminary information given by GROTHENDIECK concerning the construction of the presheaves defining the structure sheaf $\tilde{A}$ on $X = \mathrm{Spec}\, A$ and the sheaf $\tilde{M}$ of $\tilde{A}$-modules attached to an arbitrary A-module M.[1] Essentially, we are going to construct, in several ways, rings of fractions $S^{-1}A = \{a/s \mid a \in A, s \in S\}$ and modules of fractions $S^{-1}M = \{m/s \mid m \in M, s \in S\}$ with denominators in a multiplicative set $S \subset A$. We are mainly interested in the cases $S_f = \{f^n \mid n \geq 0\}$ and $S = A - \mathfrak{p}$ for $\mathfrak{p} \in \mathrm{Spec}\, A$. $S^{-1}A$, $S_f^{-1}M$ are abbreviated by the notations A_f, M_f.⌋

..... We are going to construct a sheaf of rings according to this recipe ⌈the one from Ch. II, §3⌋[2], but, at the same time we shall construct something more general, namely some sheaf of modules $\tilde{M}_x$ ($x \in X = \mathrm{Spec}\, A$). We shall start with a module M over A (which we shall eventually vary) thus to M we will associate a sheaf $\tilde{M}$.

f is not nilpotent?

Then, when M is equal to A ⌈regarded as an A-module over itself⌋, $\tilde{M}$ becomes equal to the structure sheaf $\tilde{A}$ that we are looking for.

We have to construct the functor $f \mapsto \tilde{M}(f)$ ($f \in A$). $\tilde{M}(f)$ will be the localized module M_f at the multiplicative set $S_f = \{f^n \mid n \geq 0\}$ which can be described in various ways. We now make a little parenthesis on commutative algebra.[3]

M_f can be described as being the direct limit[1] of copies of M, $M = M_\alpha$ induced by non-negative integers α, where M_α is mapped into $M_{\alpha+1}$ by multiplication by f:

$$M_\alpha \to M_{\alpha+1} \qquad \forall \alpha \geq 0 \tag{8.1}$$

---

(1) All A-modules ($A \in \mathrm{Ob}\,\mathcal{C}$) M are unitary, i.e. $(1,m) \to 1.m = m$ $\forall m \in M$, where M is an object of the category $\underline{m}_A$ of A-modules and 1 is a homomorphism of A-modules.

(2) The reader wishing to follow GROTHENDIECK's talk without my interruptions should continue with Ch. I, §3.

(3) Full details can be found in EGA, I (IHES) Ch. 0, pages 13-14,...

In this way M appears as a direct limit (cf. Ch. II) of A-modules

(8.2) $$M = \varinjlim M_\alpha$$

and therefore it is an A-module and we have, of course, a canonical homomorphism

(8.3) $$S_f : M \to M_f$$

because M can be viewed as the initial step : $M = M_0$.

So we have constructed in terms of f $(\in A)$ another module M_f. M will be fixed for a little while and f will be allowed to vary. We are going to characterize (8.3) by a universal property: Let us remark that in this module M_f multiplication by f becomes invertible (Hint: look at the limit of the transition morphisms (8.1). Moreover the map S_f of (8.3) is universal, with respect to the property that f-multiplication becomes invertible.)

Namely if $\phi : M \to N$ is an A-homomorphism such that the homothety $n \mapsto fn$ $(\forall n \in N)$ is bijective ($\Leftrightarrow$ it is an isomorphism) there exist a unique homomorphism $M_f \to N$ making the diagram

(8.4) $$\begin{tikzcd} M \arrow[r, "S_f"] \arrow[dr] & M_f \arrow[d] \\ & N \end{tikzcd}$$

commutative. So M_f appears as a solution of a universal problem (mapping M into A-modules such that the f multiplication on the image becomes an isomorphism). Therefore we see that M_f becomes functorial with respect to M, i.e. if we have a homomorphism $M \to N$ [in the category $\mathfrak{M}_A$ of A modules] then there exists a unique homomorphism $M_f \to N_f$ which makes the diagram

(8.5)
$$\begin{tikzcd} M \arrow[r] \arrow[d, "S_f"'] & M \arrow[d, "S_f"] \\ M_f \arrow[r] & M_f \end{tikzcd}$$

commutative, because the composition of $M \to M \to M_f$ has the property that the f-multiplication on the image is invertible. As a consequence S_f factors through M_f in a unique way.

So we have already two ways of describing M_f : either as a direct limit or as a solution of a universal problem. Still we can give a third description: M_f can be viewed as a set of formal fractions x/f^n

(8.6)
$$M_f = \{x/f^n \mid x \in M,\ n \geq 0\}$$

i.e. as a set of formal quotients of elements of M divided by powers of f with non-negative exponents, where the module structure (addition and multiplication by scalars) is defined in the obvious way, namely:

(8.7)
$$x/f^n + x'/f^{n'} = (f^{n'}x + f^n x')/f^{n+n'}$$

(8.8)
$$\lambda(x/f^n) = (\lambda x)/f^n \qquad \forall \lambda \in A$$

But first of all we need to define an equivalence relation between pairs $(x, f^n) \sim (x', f^{n'})$ such that $x/f^n = x'/f^{n'}$. We would like to do so by "chasing denominators." This would mean that $xf^{n'} - x'f^n = 0$. But this is not quite an equivalence relation. We have to remember that multiplication by f should be bijective on the image. This means that there should exist some $m \geq 0$ such that

(8.9)
$$f^m(xf^{n'} - x'f^n) = 0$$

Conversely (8.9) actually defines an equivalence relation, in other words:

We can define x/f^n as an equivalence class of pairs (x,f^n) $(n \geq 0)$ with respect to the equivalence relation

$$(8.10) \qquad (x,f^n) \sim (x',f^{n'}) \Leftrightarrow (8.9) \text{ holds for some } m \geq 0 .$$

We leave to the reader the easy verifications that (8.7), (8.8) are well defined.

Now we can check that multiplication by f in A_f is invertible. First of all it is surjective because we can write $x/f^n = f(x/f^{n+1})$; and it is injective also because $f.(x/f^n) = (fx)/f^n = 0$ is equivalent to the fact that there exist some $m \geq 0$ such that $f^m(fx) = f^{m+1}x = 0$ and this is equivalent to $x/f^n = 0$. So we can introduce in this way a calculus of fractions. Besides we can define now $S_f: A \to A_f$ by $x \mapsto x/1$ $(1 = f^\circ)$ or what is the same by $x \to f^n x/f^n$.

This description of M_f in terms of fractions is essentially the same as the previous one because the composition of $M = M_o \to M_1 \to \cdots\cdots\to$ maps x in $f^n x$.

When M is equal to A [regarded as an A-algebra] this constructi gives an A-module homomorphism $A \to A_f$. But A_f is not only an A-module. It is also an A-algebra, where the product is defined by

$$(8.11) \qquad (x/f^n)\ (y/f^{n'}) = (xy)/f^{n+n'} \quad {}^{(1)}$$

Furthermore, for any M , we can see that M_f is an A_f-module. This c be seen in various ways, for instance the product $(a/f^m)\ (m/f^n)$ $(a \in A, m \in$ is well defined to be equal to am/f^{n+m}. Another way of seeing this is a

(1) The reader should verify that the product is well-defined (independent of representatives and that the ring axioms are satisfied).

follows: The ring A_f is an A-algebra characterized by the following universal property:

Any ring homomorphism $A \xrightarrow{\varphi} R$ in a ring with unit (not necessarily commutative) such that $\varphi(f)$ is invertible factors through A_f, making commutative the diagram

(8.12)
$$\begin{tikzcd} A \arrow[r] \arrow[dr] & A_f \arrow[d, dashed] \\ & R \end{tikzcd}$$

Now if we take R to be the endomorphism ring End (N) of an Abelian group to give a homomorphism $A \to R$ is the same thing as to define in N a module structure over A. If this structure has the property that f operates as an isomorphism on N this means that N has the structure of an A_f-module and conversity: To give an A_f-module structure over N is the same as to give a structure of an A-module on N such that f operates as an isomorphism. If we apply this to the particular case of M_f then the A-module M_f becomes an A_f-module because f operates on M_f as an isomorphism.

Finally we obtained several ways of describing the A_f-module structure on M_f. Now we shall see what happens for a variable $f \in A$.

Now let us assume

(8.13) $$X_f \subset X_g$$

and let us define an A-homomorphism

(8.14) $$M_g \to M_f$$

We shall see that there exists a unique A-homomorphism making

(8.15)
$$\begin{tikzcd} M \arrow[r] \arrow[d] & M \arrow[d] \\ M_g \arrow[r] & M_f \end{tikzcd}$$

commutative (where the vertical arrows are the canonical homomorphisms of M into the localized modules M_g, M_f). All I need to prove is that multiplication by g operating on M_f is an isomorphism (thus it factors through M_g). In fact, because of the inclusion (8.13) we have an identity of type $f^n = gh$ ($n \geq o$, $h \in A$) and since f becomes invertible in M_f, g becomes invertible in M_f.

On the other hand this map $M_g \to M_f$ (when (8.14) holds for f, g fixed) is functorial with respect to M. Because of the uniqueness of the induced map (8.14) we have also <u>transitivity</u>: If

(8.16) $$X_f \subset X_g \subset X_n$$

we have a commutative diagram

(8.17) $$\begin{tikzcd} M_n \arrow[r] \arrow[dr, dotted] & M_g \arrow[d] \\ & M_f \end{tikzcd}$$

because the unique dotted arrow should be equal to the composition.

[In order to interpret correctly all the details of the construction of the sheaves $\tilde{A}$, $\tilde{N}$ in §9 we need to also know the properties of <u>localizations or modules or rings of fractions with respect to arbitrary multiplicative sets $S \subset A$</u>. This construction of commutative algebra can be regarded as a wide generalization of the construction of the field of fractions Ω of an integral domain I. Let us write $I^* = I - \{0\}$. Then, a fraction $a/b \in \Omega$ ($a \in I$, $b \in I^*$) is an equivalence class of $I \times I^*$ with respect to the equivalence relation

(8.18) $$(a_1, b_1) \sim (a_2, b_2) \Leftrightarrow a_1 b_2 - a_2 b_1 = 0$$

There is a canonical map $I \to \Omega$ defined by

(8.19) $\qquad a \mapsto a/1$

If $A \in \mathrm{Ob}\mathcal{C}$ but is not an integral domain the corresponding property (8.18) (equality of cross products) does not define any equivalence relation. However it is possible to define such a generalization of the "calculus with fractions" just by replacing I^* by an arbitrary multiplicative set S of A, (already found in the definition of prime ideals (cf. Ch. I).) We recall the definition:

A subset S of A ($A \in \mathrm{Ob}\mathcal{C}$) is called multiplicatively closed iff the two following properties hold

1) $\quad (a,b) \in S \times S \Rightarrow ab \in S$

2) $\quad 1 \in S$

1), 2) can be replaced by the unique condition that the product of any finite family of elements of S belongs to S. Then the unit element of A appears to be the product of an empty family of elements of S.[1]

Such sets always exist in any $A \in \mathrm{Ob}\,\mathcal{C}$ for instance we can take $S = A$. Non trivial examples are:

1) $S = A - \mathfrak{p}$ if $\mathfrak{p} \in \mathrm{Spec}\, A$ (by definition of prime ideal $\mathfrak{p}$ prime $\Leftrightarrow$ $A - \mathfrak{p}$ mult.).

2) The set of non-zero divisors of any $A \in \mathrm{Ob}\mathcal{C}$ is multiplicative.

3) Let f be any element of A. Then

(8.20) $$S_f = \{f^n \mid n \in \mathbb{Z},\, n \geq 0\}$$

---

(1) If J is a finite non empty set and $\varphi(J)$ a family of elements of A indexed by J such a family is a map $\varphi: J \to A$. Then $\Pi_\varphi = \prod_{j \in J} \varphi(j)$. If φ_1 and φ_2 are two finite non empty families $\Pi_{\varphi_1+\varphi_2} = \Pi_{\varphi_1} \cdot \Pi_{\varphi_2}$: If we want to maintain this property for $J = \emptyset$, we note there is a unique map $\emptyset \overset{i}{\hookrightarrow} A$ and $\varphi + i = \varphi$ thus $\Pi_\emptyset = 1$.

is a multiplicative set. For $f = 0,1$. $S_0 = \{0\}$, $S_1 = \{1\}$.

The original case, when A is an integral domain becomes a particular case because A is an integral domain iff $\{0\} \in \operatorname{Spec} A$.

The intersection of any non empty family of multiplicative sets (m.s.) of A is multiplicative. In particular for any $T \subset A$, the intersection of the (non empty!) family of all m.s. of A containing T is the smallest m.s. S_T containing T . S_T consists of all the products of finite families of elements of T .

The interest of the multiplicative sets comes from the following proper

Let S be any m.s. of A . Let $(a_i, s_i) \in A \times S$, $i = 1,2$ be two pairs. Let us write $(a_1,s_1) \sim (a_2,s_2)$ iff

$$s(a_1 s_2 - a_2 s_1) = 0 \tag{8.21}$$

for some $s \in S$. Then $\sim$ is an equivalence relation.[1] In the particular case that A is integral and $S = A - \{0\}$, (8.21) becomes (8.18).

We denote by $S^{-1}A$ the quotient set $A \times S/\sim$ in order to recall that $S^{-1}A$ consists of "fractions" a/s with numerators in A and denominators in S . Precisely a/s denotes the $(\sim)$-class of $(a,s) \in A \times S$.

$S^{-1}A$ has a natural structure of commutative ring with unit defined by the usual laws $a_1/s_1 + a_2/s_2 = (a_1 s_2 + a_2 s_1)/s_1 s_2$ and $(a_1/s_1)(a_2/s_2) = (a_1 a_2)/s_1 s_2$; $0/1$ and $1/1$ are the zero and unit element of $S^{-1}A$.

There is a natural morphism in $\mathcal{C}$: $A \to S^{-1}A$ defined by (7.2) that will be denoted by $/1_S$ or just $/1$ when S is fixed.

[1] Reflexivity and symmetry are clear. Transitivity follows from: $(a,s) \sim (a',s')$ $\Leftrightarrow t(as' - a's) = 0$ $(a',s') \sim (a'',s'')$ $\Leftrightarrow t'(a's'' - a''s')$ $(t,t' \in S)$ and $tt's'(a''s - s''a) = 0$ with $tt's' \in S$.

WARNING: We cannot write "naturally" $a = a/1$, because for an arbitrary S $/1$ is not necessarily injective. In fact, we can compute easily the kernel of $/1_S$: $a/1 = 0/1$ iff $sa = 0$ for some $s \in S$. As a consequence: $/1_S$ is injective iff S does not contain divisors of zero.

$S^{-1}A$ is called the ring of fractions of S with denominators in S . In particular if S is the set of all non zero divisors of A , $S^{-1}A$ is called the total ring of fractions of A (it is not necessarily a field). Since $/1_S$ is injective in this case we can embed A in $S^{-1}A$.

The ring of fractions of A with respect to the multiplicative set $S_f = \{f^n | n \geq 0\}$ plays a very important role in the construction of the structure sheaf $\tilde{A}$ of the affine scheme (Spec A, $\tilde{A}$) attached to any $A \in G$.

Both $S^{-1}A$ and the morphism $/1_S$: $A \to S^{-1}A$ are called localizations although $S^{-1}A$ is referred to as the ring A localized at S .

This localization morphism $/1_S$ has the following universal property

The image $s/1$ of any element $s \in S$ is invertible in $S^{-1}A$; precisely $(s/1)(1/s) = 1/1$. Moreover, let φ: $A \to B$ be any morphism in G such that $\varphi(s)$ is invertible in B for any $s \in S$. Then, there exist a unique morphism u: $S^{-1}A \to B$ such that the diagram

(8.19)
$$\begin{tikzcd} A \arrow[r, "/1_S"] \arrow[dr, "\varphi"'] & S^{-1}A \arrow[d, dashed, "u"] \\ & B \end{tikzcd}$$

commutes.

Let $S \hookrightarrow T$ be an inclusion between two multiplicative sets in A . The induced inclusion $A \times S \hookrightarrow A \times T$ factors through the equivalence relation.

$\sim_S$ and $\sim_T$ thus there is an induced homomorphism $S^{-1}A \to T^{-1}A$.

A particularly important special case needed in the construction of $\tilde{A}$ is the case that $T \supset S$ is the saturated multiplicative set $\overline{S}$ deduced from S . Its definition is .

$$\overline{S} = \{a \in A \mid \exists s \in S \text{ s.t. } s = ab,\ b \in A\}$$

in other words $\overline{S}$ consists of the divisors of elements of S .(1)

The interest of $\overline{S}$ is that the canonical homomorphism $S^{-1}A \to \overline{S}^{-1}A$ is an isomorphism. Besides $\overline{\overline{S}} = S$.

In fact in any $S^{-1}A$ the rule $a/s = at/st$ (for any $t \in S$) is valid Regarding division, if $a/\overline{s} \in \overline{S}^{-1}A$ there exist $t \in \overline{S}$ such that $\overline{s}t \in S \Rightarrow a/\overline{s} = at/\overline{s}t \in S^{-1}A$, thus $S^{-1}A \cong \overline{S}^{-1}A$, q.e.d.

EXAMPLES: For every $\mathfrak{p} \in \operatorname{Spec} A$, the localized ring $(A - \mathfrak{p})^{-1}A$ is denoted more simply by $A_{\mathfrak{p}}$ and it is commonly referred to as the localization of A at $\mathfrak{p}$. In particular if $(0) \in A$ ($\Leftrightarrow$ A is an integral domain) $A_{(0)}$ is the field of fractions of A .

The name is justified because $A_{\mathfrak{p}} \in \mathfrak{C}$ is in fact a local ring. ⌈We already encountered local rings through our discussion of ringed spaces and locally ringed spaces in §7. I will give a brief review of the technical algebraic treatment of local rings, following the tape: ⌋

DIGRESSION ON LOCAL RINGS(2)

..Let $\mathcal{O}$ be a ring (commutative with unit, as usual, $\Leftrightarrow \mathcal{O} \in \operatorname{Ob} \mathfrak{C}$.

(1) Check the immediate facts that $\overline{S}$ is multiplicatively closed and contains S .

(2) ⌈We advise the knowledgeable reader to skip this digression if he is mainly interested in rings of fractions and localizations.⌋

We say that <u>$\mathcal{O}$ is a local ring</u>[1] <u>if and only if there is one and only one maximal ideal m in $\mathcal{O}$</u>. We recall the fact that the existence of at least one maximal ideal implies that $\mathcal{O}$ is not zero (because of KRULL's Theorem). In terms of the Spectrum, if you like geometrical language we know that maximal ideals of A correspond bijectively to closed points of Spec A. Thus <u>$\mathcal{O}$ is a local ring iff there is just one closed point in Spec $\mathcal{O}$</u>.

Let $(\mathcal{O}, m)$ be a pair where $\mathcal{O} \in \mathrm{Ob}\,\mathcal{C}$, and m is an ideal of $\mathcal{O}$. <u>$\mathcal{O}$ is local with maximal ideal m iff</u>

$$\mathcal{O}^* = \mathcal{O} - m^{(2)}$$

This comes from the fact that <u>in any ring</u> A [$\in \mathrm{Ob}\,\mathcal{C}$] (<u>not</u> necessarily local) and element f is invertible if and only if it does not belong to any maximal ideal, thus $fA = A$ is the unit ideal, because we have: $A/f0 = 0$ (because of KRULL's theorem) $\Rightarrow A = fA$. Conversely if $\mathcal{O}^* = \mathcal{O} - m$ the quotient ring $\mathcal{O}/m$ is a field because any element of $\mathcal{O}/m \neq 0$ is the image of some invertible $f \in \mathcal{O}^*$. Any ideal $\mathfrak{a} = \mathcal{O}$ of $\mathcal{O}$ is contained in m otherwise if $f \in m$, $f \notin m$ f is invertible $\Rightarrow \mathfrak{a} = \mathcal{O}$.

<u>The field</u> $k(m) = \mathcal{O}/m$ <u>is called the residue field of the local ring $\mathcal{O}$</u>.

---

(1) It is frequent to include the Noetherian condition in the definition. If for instance NAGATA: $\mathcal{O}$ local $\Leftrightarrow$ ($\mathcal{O}$ Noetherian + $\mathcal{O}^* = \mathcal{O}^* - m$), then the second condition alone is referred to as $\mathcal{O}$ quasi-local.

GROTHENDIECK has a certain tendency to get rid of Noetherian restrictions. For instance in our context for an arbitrary A, $A_{\mathfrak{p}}$ is not necessarily Noetherian. Thus when $A_{\mathfrak{p}}$ is Noetherian we shall make it explicit by talking about <u>Noetherian local rings</u>.

(2) Another characteristic condition for $\mathcal{O}$ to be local is the following one whose proof is left to the reader: $\mathcal{O}$ [$\in \mathrm{Ob}\,\mathcal{C}$] is local iff for every $f \in \mathcal{O}$ <u>either</u> f <u>or</u> $1 - f$ <u>is invertible</u>.

As a natural example let us check that $A_{\mathfrak{p}}$ is in fact a local ring with maximal ideal $\mathfrak{m} = \mathfrak{p} A_{\mathfrak{p}}$[1] for every $\mathfrak{p} \in \operatorname{Spec} A$:

An element a of $A_{\mathfrak{p}}$ is a fraction f/g $(f, g \in A,\ g \notin \mathfrak{p})$. a be. to $\mathfrak{p}A_{\mathfrak{p}}$ iff it can be written as f/g with $f \in \mathfrak{p}$ thus $f/g = f(1/g)$. $\mathfrak{p}A_{\mathfrak{p}}$ is clearly an ideal of $A_{\mathfrak{p}}$. The element a does not belong to $\mathfrak{p} A_{\mathfrak{p}}$ iff it cannot be written in this form, i.e. iff $a = f/g$ with $f \notin \mathfrak{p}$, $g \notin \mathfrak{p}$. Then a is invertible with $a^{-1} = g/f$.

9. DEFINITION OF THE SHEAVES $\tilde{M}$, $\tilde{A}$.

...So we have a presheaf $\tilde{M}$ (or $\tilde{A}$) of A-modules (of A-algebras) defined in the ordered category of elements $f \in A$:

(9.1)

$$f \mapsto M_f \quad (\text{or } f \mapsto A_f) \quad \forall f \in A$$

We could in fact take the associated sheaf as definition of $\tilde{M}$, $\tilde{A}$ as we did in the book of DIEUDONNÉ (EGA-I-Springer, DEF 1-3-4, page 196) and be happy..., but we want to prove directly that the presheaves $\tilde{A}$, $\tilde{M}$ are sheaves[2]: i.e. for any open covering

(9.2)

$$X_f = \bigcup_{i \in I} X_{f_i}$$

of basic open sets in Spec A we can define by functoriality the diagrams

(9.3)

$$0 \to M_f \xrightarrow{\Pi\rho_i} \prod_{i \in I} M_{f_i} \overset{\Pi\rho_{i,j}}{\rightrightarrows} \prod_{i,j} M_{f_i f_j} \to 0$$

(9.3)'

$$0 \to A_f \xrightarrow{\Pi\rho_i} \prod_{i \in I} A_{f_i} \overset{\Pi\rho_{i,j}}{\rightrightarrows} \prod_{i,j} A_{f_i f_j} \to 0$$

and we need to check that they are exact, for any $f \in A$ and for any covering of X_f.

In this course (and in most of his publications) GROTHENDIECK takes the exactness condition as definition of a sheaf, because it is more categorical, i.e. we can replace Op(X) by any category with fiber products; summarizing: a sheaf over X is a presheaf satisfying the exactness condition.

With this approach the "stalks" do not appear explicitly but this is convenient for arbitrary "sites" (= GROTHENDIECK's Topologies) where the stalks do not play any primary role.

[1] The localization map $A \to A_{\mathfrak{p}}$ gives $A_{\mathfrak{p}}$ a canonical structure of A-algebra, thus if $B \subset A$, the set $BA_{\mathfrak{p}} \subset A_{\mathfrak{p}}$ makes sense.

[2] We proves something more here: $\tilde{M}$, $\tilde{A}$ are sheaves over the site $\mathcal{B}$ of basic open sets of the spectral topology in Spec A, cf. Ch. II §3.

Let $U \in \mathrm{Ob}\mathcal{B}$ be any basic open set of $X = \mathrm{Spec}\, A$. Then, by definition there is at least one element $f \in A$ such that $U = X_f$, but this f does not need to be unique, for instance $X_f = X_{f^n}$ $(n \geq 1)$ and we can find plenty of examples where $f \neq f^n$. The <u>modules and rings M_f, A_f apparently depending on the choice of a representative f depends actually on $U (= X_f)$ only</u>, because the multiplicative set $S_f = \{f^n | n \geq 0\}$ defines the same ring as the saturated multiplicative set S_U which depends only on U. In other words any fraction $m/s \in S_U^{-1}M$ (or $a/s \in S_U^{-1}A$) can be written also in the form m'/f^n (if $U = X_f$) or a'/f^n. As a consequence we can replace (9.1) by

(9.1)' $$U \to \tilde{M}(U) \qquad (\text{or } U \to \tilde{A}(U)) \qquad \forall U \in \mathrm{Ob}\mathcal{B}$$

where we wrote $M(U)$, $A(U)$ for short instead of $S_U^{-1}M$, $S_U^{-1}A$ and replace (9.2) by

(9.2)' $$U = \bigcup_{i \in I} U_i$$

with $U_i = X_{f_i}$ in such a way that the transition morphisms $M_f \to M_{fi}$ $(A_f \to A_{f_i})$ can be written intrinsically as $M(U) \to M(U_i)$ $(A(U) \to A(U_i))$, i.e. they are independent on the choice of f, f_i. In other words the functorial properties of $\tilde{M}$, $\tilde{A}$ are indeed independent of any choice of representatives. In §3 we completed the necessary steps in order to make these variations very easily.

In order to prove that $\tilde{M}$ is a sheaf I need to prove that for every $f \in A$ the diagram (9.3) is exact. Since X is quasi-compact we will assume I to be <u>finite</u>. We shall divide the proof in two natural steps.

1) $\Pi\rho_i$ is injective ($\Leftrightarrow$ exactness in the first step)

Let us assume that all the images $\rho_i(x/f^n)$ are zero for every $i \in$ where $x/f^n \in M_f$. We need to prove that $x/f^n = 0$.

If $\rho_i(x/f^n) = (x/f^n)/1 \in M_{f_i}$ is equal to zero there exists some $m_i \geq 0$ such that $f_i^{m_i}(x/f^n) = (f_i^{m_i}x)/f^n = 0$ (in M_f) and this implies the existence of some $f^{m_i'}$ such that $f^{m_i'}f_i^{m_i}x = 0$; then since I is finite we can replace m_i' by $m = \max m_i'$ $(i \in I)$ and we obtain

$$f^m f_i^{m_i} x = 0 \tag{9.4}$$

On the other hand the inclusion $X_f \supset \bigcup_{i \in I} X_{f_i^{m_i}}$ implies the existence of a relation $f^{m'} = \sum_{i \in I} g_i f_i^{m_i}$ that together with (9.4) implies $f^{m+m'}x = 0 \Rightarrow x/f^n$ is equal to zero in M_f, q.e.d. This takes care the exactness of (9.3) in the first step.

2) Exactness in the middle term of (9.3):

Let us assume the matching condition

$$\rho^{f_i}_{f_i f_j}(x_i/f_i^{n_i}) = \rho^{f_j}_{f_i f_j}(x_j/f_j^{n_j}) \in M_{f_i f_j} \tag{9.5}$$

for every pair $i,j \in I$ where $x_i/f_i^{n_i} \in M_{f_i}$, $\forall i \in I$. Since I is finite we can assume $n = n_i$, $\forall i$ just multiplying x_i and $f_i^{n_i}$ by suitable powers of f_i. Moreover since $M_{f_i} = M_{f_i^n}$ $(\forall n > 1)$ we can assume $n = 1$. Then the matching condition assumes the form

$$(f_i f_j)^{m_{ij}}(f_j x_i - f_i x_j) = 0 \qquad m_{ij} \text{ integers } \geq 0 \tag{9.6}$$

and again we can replace the (finitely many) m_{ij} by some fixed $m > 0$:

(9.6)'
$$(f_i f_j)^m (f_j x_i - f_i x_j) = 0$$

or

(9.6)"
$$f_j^{m+1} y_i - f_i^{m+1} y_j = 0$$

where $y_i = f_i^m x_i$ $(i \in I)$ (i.e. $x_i/f_i = y_i/f_i^{m+1}$), thus (9.6)' reduces to the case $m = 1$.

Let us write now $y = \sum_{i\in I} g_i y_i$ (where, as before $f^n = \sum_{i\in I} g_i f_i^{m+1}$).

Then we have

(9.7)
$$f^n y_i = (\sum_{j\in I} g_j f_j^{m+1}) y_i = f_i^{m+1} (\sum_{j\in J} g_j y_j) = f_i^{m+1} y$$

which is equivalent to the fact that $\rho_i(y/f^n) = y_i/f_i^{m+1}$ with $y/f^n \in M_f$ q.e.d.

REMARK. The previous construction shows that localization of an A-module M with respect to a variable element $f \in A$ can also be used as a localization in the topological sense, namely as taking sections of a certain sheaf $\tilde{M}$ with respect to a smaller basic open set X_g [$\in Ob\beta$].

In particular, we have

(9.8)
$$\tilde{M}(X) = M_1 = M$$

[<u>Localization with respect to the unit element of</u> A]

Thus <u>we view M as the A-module of global continuous sections of the sheaf $\tilde{M}$ over Spec A</u> [i.e. we "recover" M] and the localization map

$M = M_1 \to M_f$ is interpreted as the restriction in the topological sense t the basic open set X_f.(1)

In the case $M = A$:

$\tilde{A}(X) = \tilde{A}_1 = A$ i.e. <u>the original ring A becomes isomorphic with this ring of global continuous sections of the structure sheaf</u> $\tilde{A} = \mathcal{O}_X$ <u>on</u>

This interpretation of sheaf theory over Spec A is very importa if we want to give any geometric sense to commutative algebra .

10. IDENTIFICATION OF $\mathbb{G}^*$ WITH THE CATEGORY OF AFFINE SCHEMES.

We know that the category $\mathbb{C}^*$ opposite to the category $\mathbb{G}$ of commutative rings with unit is isomorphic with the category Aff of <u>affine algebraic spaces</u> (cf. Ch. I) or what is the same, the category of covariant representable functors: $\mathbb{G} \to$ Sets. Namely to every $A (\in \mathrm{Ob}\,\mathbb{G}$ corresponds the affine algebraic space $\Xi_A = V_A$, i.e. the representable functor $k \to \mathrm{Hom}_{\mathbb{G}}(A,k)$ $(k \in \mathrm{Ob}\,\mathbb{G})$. GROTHENDIECK's second (and more <u>geometric</u>) interpretation of $\mathbb{G}^*$ in terms of <u>affine schemes</u> is as follows:

To every $A \in \mathrm{Ob}\,\mathbb{G}$ corresponds the <u>affine scheme</u> (Spec $A,\tilde{A}$) , i.e. the <u>locally ringed spa</u> whose underlying topological space is the spectrum Spec $A = \{\mathfrak{p} \mid \mathfrak{p}$ prime in $A\}$ endowed with its spectral topology (cf. §2) and $\tilde{A}$ is the structure sheaf defined in §9. Conversely (Spec $A,\tilde{A}$) gives back $A = \Gamma(\mathrm{Spec}\,A,\tilde{A})$ as the ring of global sections of the structure sheaf An arbitrary ring homomorphism in $\mathbb{G}$,

(10.1)

$$f: A \to B$$

induces a morphism

(10.2)

$$f^*: (\mathrm{Spec}\,B,\tilde{B}) \to (\mathrm{Spec}\,A,\tilde{A})$$

<u>in the category $\mathcal{L}rs$ of locally ringed spaces</u> (cf. §7), where $f^* = (\mathrm{Spec}\,f,\tilde{f})$ consists of the continuous map

(10.3)

$$\mathrm{Spec}\,f: \mathrm{Spec}\,B \to \mathrm{Spec}\,A$$

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(1) This informal description can be made more precise, following EGA-Springer, (Prop. 1.3.6, page 199) by the following statement:
<u>The open set</u> X_f <u>can be identified canonically with the Spectrum</u> Spec A_f <u>of the localized ring</u> A_f <u>and</u> $\tilde{M}_f$ <u>with the restriction</u> $\tilde{M}|X_f$.

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induced between the underlying topological spaces and a (Spec f)-morphism (cf. §):

(10.4) $$\tilde{f}: \tilde{A} \to \tilde{B}$$

among the structure sheaves.

Conversely: any morphism $G = (\varphi,\theta)$: $(\mathrm{Spec}\, B, \tilde{B}) \to (\mathrm{Spec}\, A, \tilde{A})$ between the two l.r.s. induces a morphism $G^*: A \to B$ among the rings of global sections of both structure sheaves such that

(10.5) $$\mathrm{Spec}\, G^* = \varphi \qquad \tilde{G}^* = \theta$$

In other words:

<u>The affine schemes form a full subcategory of the category</u> $\mathfrak{Lrs}$ <u>of locally ringed spaces that we can identify with the category</u> $\mathcal{G}^\circ$ <u>opposite to</u> $\mathcal{G}$.

The two functors $f \to (\mathrm{Spec}\, f, \tilde{f})$ $((\varphi, \theta \to \theta^*))$ are contravariant functors $\mathcal{G} \to \mathcal{G}^\circ$ $(\mathcal{G}^\circ \to \mathcal{G})$ which <u>establish a complete dictionary between the language of commutative algebra and the geometric language of locally ringed spaces.</u>

We have seen that if we associate to every commutative ring with unit A the ringed space $(X,\mathcal{O}_X)$:

(10.6) $$A \mapsto (X,\mathcal{O}_X) \qquad X = \mathrm{Spec}\, A,\ \mathcal{O}_X = \tilde{A}$$

We can recover the ring A as being the ring of global sections of the structure sheaf $[A \cong \Gamma(X,\mathcal{O}_X)]$ therefore we can feel very secure that we have recaptured all information regarding A (or what amounts to the same, about the corresponding algebraic space $\mathfrak{X}_A$ [cf. Ch. II] in terms of this geometric object [the <u>affine scheme</u> $(\mathrm{Spec}\, A,\tilde{A})$] . But in order to feel still more at ease we have to see how the morphism between rings [in $\mathcal{G}$] (or equivalently between affine algebraic spaces) can be interpreted in terms of the spectra. So let us now give a homomorphism

(10.7) $$u: A \to B$$

[in $\mathcal{G}$] and look at the corresponding map between the spectra.

(10.8) $$\varphi: X \to Y$$

(where $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$ and $\varphi = \operatorname{Spec} u = u^*$) which carries every prime ideal p of B into its inverse image $u^{-1}(p) \in X$. We have seen [cf. §4] that this is a continuous map from X into Y.

Now I want to see how this map relates to the sheaves on Y and X associated to modules over A and B. Let us start with the case of the structure sheaves $\mathcal{O}_X, \mathcal{O}_Y$. The map φ induces a natural homomorphism of sheaves of rings

(10.9) $$\tilde{\varphi} : \mathcal{O}_Y \to \mathcal{O}_X$$

(in the opposite direction of φ), or what amounts to the same we have a natural homomorphism of the sheaf of rings $\mathcal{O}_Y$ into the direct image $\varphi_*(\mathcal{O}_X)$ of $\mathcal{O}_X$ (2) by φ. In order to define $\tilde{\varphi}$ (cf. (10.9)) I need to define it only for the presheaves, namely for every $f \in A$ I have to define a ring homomorphism:

(10.10) $$\Gamma(Y_f,\mathcal{O}_y) \to \Gamma(\varphi^{-1}(Y_f),\mathcal{O}_X)^{(1)}$$

between the sections of both sheaves over the open sets $Y_f \subset Y$ and $\varphi^{-1}(Y_f) \subset X$. Now we need to make explicit what (10.10) means, using the fact that $(Y_f,\mathcal{O}_y) = A_f$ (A localized in f cf. §) and the formula $\varphi^{-1}(Y_f) = X_{u(f)}$ (proved in §4), which implies $\Gamma(\varphi^{-1}(Y_f)|\mathcal{O}_X) = B_{u(f)}$, or,

(1) Remember that $\Gamma(\varphi^{-1}(Y_f),\mathcal{O}_X)$ is the ring of sections over Y_f of the direct image $\varphi_*(\mathcal{O}_X)$ of $\mathcal{O}_*$ in Y [cf. §7].

what is the same, we have a homomorphism

(10.11) $$A_f \to B_{u(f)} (= B_f)$$

induced by u. These homomorphisms are compatible with transition maps, i.e. if $X_g \hookrightarrow X_f$ we have a commutative diagram

(10.12)
$$\begin{tikzcd} A_f \arrow[r] \arrow[d] & B_{u(f)} = B_f \arrow[d] \\ A_g \arrow[r] & B_{u(g)} = B_g \end{tikzcd}$$

where the horizontal arrows are of type (10.11), i.e. $x/\overline{f^n} \mapsto u(x)/\overline{(u(f))^n}$ and the vertical arrows denote restriction morphisms. This means that φ can be viewed as a morphism of ringed spaces[1] [cf. §) D: $(X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$.

REMARK. This homomorphism $A \overset{u}{\to} B$ can be reconstructed when we know the corresponding homomorphism for the ringed spaces $(X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$, because u can be viewed as being the homomorphism $\Gamma(Y,\mathcal{O}_Y) \to \Gamma(X,\mathcal{O}_X)$ induced between the two rings of global sections it suffices to take $f = 1 \Rightarrow u(f) = 1,\ y = y_1 \mapsto x = x_1$. In other words we have seen quite trivially that ring homomorphisms can be interpreted as particular cases of morphisms of ringed spaces, i.e. we have an injective map:

(10.13) $$\mathrm{Hom}_{\mathcal{G}}(A,B) \hookrightarrow \mathrm{Hom}_{Rs}((X,\mathcal{O}_X),(Y,\mathcal{O}_Y))$$

Now we want to see what are the special properties of the morphisms in the category Rs of ringed spaces which express that they come from ring homomorphisms. Here we have to recall that both $(X,\mathcal{O}_X),(Y,\mathcal{O}_Y)$ are objects of the category $\mathcal{L}rs$ of locally ringed spaces, i.e. that the

(1) The equality $B_{u(f)} = B_f$ recalls the fact that the localization of the ring B [$\in \mathrm{Ob}\,\mathcal{G}$] in the image $u(f)$ of f by u is canonically isomorphic with the localization of B regarded as an A-module with respect to f. We leave the easy proof to the reader.

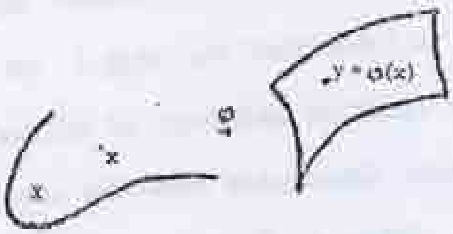
<u>stalks of the structure sheaves at every point</u> $x \in X$ (or $y \in Y$) <u>are local rings</u>. [In other words (10.13) can also be written as

(10.14) $$\mathrm{Hom}_{\mathfrak{L}}(A,B) \leftrightarrow \mathrm{Hom}_{\mathfrak{Lrs}}((X,\mathcal{O}_X),(Y,\mathcal{O}_Y))]$$

But a homomorphism $\varphi: X \to Y$ of ringed spaces implies that for every choice of $x \in X$, $y \in Y$ such that $y = \varphi(x)$ we got a ring homomorphism

(10.15) $$u_x: \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$$

in the opposite direction between the stalk at y, x. This homomorphism



u_x [in $\mathfrak{L}$] between the two local rings is kind of "reasonable" iff it is a so-called <u>local homomorphism</u>, i.e. iff the inverse image $u^{-1}(m_x)$ of the maximal ideal $m_y \subset \mathcal{O}_{Y,y}$ i.e.

(10.16) $$u_x \text{ local } \Leftrightarrow u^{-1}(m_x) = m_y \text{ .}^{(1)}$$

Accordingly, a morphism $\varphi: (X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$ [in $\mathfrak{Rs}$] of ringed spaces is said to be <u>local</u>, or a morphism of <u>locally ringed spaces</u> [= in $\mathfrak{Lrs}$]

(1) [Remember that for an arbitrary u (in $\mathfrak{L}$), $u: \mathcal{O} \to \mathcal{O}'$ ($\mathcal{O},\mathcal{O}'$ local rings) we have $u^{-1}(m') \Leftarrow m$. u is said to be <u>local</u> iff this inclusion becomes equality: $u^{-1}(m') = m$ and this is equivalent to $u(m) \subset m'$ (cf. $\frac{1}{4}$ page) I continue with the tape] ...this condition insures that u passes to the quotient $k \to k'$ (where k,k' are the residue fields $k = \mathcal{O}/m$, $k' = \mathcal{O}'/m'$) in such a way that the diagram

$$\begin{array}{ccc} \mathcal{O} & \to & \mathcal{O}' \\ \downarrow & & \downarrow \\ k & \to & k' \end{array}$$

commutes (where the vertical bars are canonical projections). Another way of viewing this is the following: $f \in \mathcal{O}^* \Leftrightarrow u(f) \in \mathcal{O}'^*$. Of course if f is invertible in $\mathcal{O}$ [$\Leftrightarrow f \in \mathcal{O}^*$], $u(f)$ is also invertible in $\mathcal{O}'^*$ (whether or not $\mathcal{O},\mathcal{O}'$ are local). But the opposite implication $\Leftarrow$ holds only the local case.

iff for every pair of corresponding points $(x, \mathcal{O}(x))$ the homomorphism (10.15) between the stalks is a <u>local homomorphism of local rings</u>.(1) Now we can prove that the image of $\mathrm{Hom}_{\mathbb{G}}(A,B)$ in the map (10.4) is a set of local morphisms. The converse is true, thus the inclusion (10.4) is actually a bijection

$$(10.16) \qquad \mathrm{Hom}_{\mathbb{G}}(A,B) \overset{\sim}{\to} \mathrm{Hom}_{\mathfrak{L}rs}((\mathrm{Spec}\ B, \tilde{B}), (\mathrm{Spec}\ A, \tilde{A})) = \mathrm{Mor}_{\mathfrak{L}rs}(X,Y)$$

11. RECOVERY OF THE LOST GROUND RING k. The dictionary we have set up so far between rings and affine schemes viewed as particular cases of locally ringed spaces, tells us how to interpret the ring A of $\mathrm{Ob}\,\mathbb{G}$, or, if you prefer the object $\mathfrak{X}_A$ of the opposite category of affine algebraic spaces which are those in which we were interested from the very start (cf. Ch. II) in terms of the l.r.s. $(\mathrm{Spec}\ A, \tilde{A})$, i.e. in terms of the <u>affine scheme</u> bijectively attached to the affine algebraic space $\mathfrak{X}_A$ $(\in \mathrm{Ob}\ \mathrm{Aff}_{\mathbb{Z}})$(1).

Using precise categorical language we have the functors

$$(11.1) \qquad \mathrm{Sch}\colon \mathbb{G} \to \mathfrak{L}rs \qquad \mathrm{Sch}^{*}\colon \mathrm{Aff}_{\mathbb{Z}}^{(2)} \to \mathfrak{L}rs$$

$\mathrm{Sch} = (\mathrm{Spec}, \tilde{\ })$ <u>contravariant</u>, Sch^{*} <u>covariant</u>.

Both images $\mathrm{Sch}(\mathbb{G})$, $\mathrm{Sch}^{*}(\mathrm{Aff}_{\mathbb{Z}})$ can be identified with a full sub-

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(1) $\mathrm{Aff}_{\mathbb{Z}}$ denotes the category of (<u>absolute</u>, i.e. over $\mathbb{Z}$) affine algebraic spaces. The category Aff_k, needed very soon, is the category of affine algebraic spaces over k $(k \in \mathrm{Ob}\,\mathbb{G})$.

(2) Where $\mathrm{Aff}_{\mathbb{Z}}$ has been canonically identified with the opposite category $\mathbb{G}^{*}$ of $\mathbb{G}$. Remember that $\mathbb{G} = \mathbb{G}_{\mathbb{Z}}$ (cf. Ch. I, § x), i.e. rings of $\mathbb{G}$ and $\mathbb{Z}$-algebras are the same thing.

category of $\mathfrak{Lrs}$, i.e. both Sch, Sch° are <u>fully faithful</u> (with reversin arrows, of course, for Sch).

Our ground ring k ($\in$ Ob $\mathfrak{A}$) has been "lost" somewhat because it did not play any role in the constructions of Sch(A) . So we need to see ho to recover it. Let

(11.2)
$$u: k \to A$$

be the structure morphism of the k-algebra A . Then our dictionaries (11. give us two maps:

(11.3) $\quad$ Schu: Sch(A) $\to$ Sch(k) $\qquad$ $\mathrm{Sch}^{\circ}(\mathfrak{X}_k) \to \mathrm{Sch}^{\circ}(\mathfrak{X}_A)$

(the first one reversing arrows). By decomposing Sch u in (Spec u, $\tilde{u}$) we see that $\tilde{u}: \tilde{k} \to \tilde{A}$ gives the sheaf-homomorphism between the structure sheaves; looking at the stalks in corresponding points we have homomorphism $k \to \mathcal{O}_x \to \mathcal{O}_y$ which, by composition tells us that <u>every stalk</u> of $\tilde{A}$ has a induced structure of k-algebra. Since $A \approx \Gamma(X,\tilde{A})$ the converse property is true; any structure of $\tilde{A}$ as a sheaf of k-algebras yields a structure morphism (1.2) of A as a k-algebra.

We can summarize these remarks as follows:

If we reintroduce $k \in \mathrm{Ob}\,\mathfrak{A}$ as a ground ring, $\mathfrak{A}$, $\mathfrak{Lrs}$, Aff, Sch are replaced by

(11.1) $\quad$ $\mathrm{Sch}_k: \mathfrak{A}_k \to \mathfrak{Lrs}_k \qquad \mathrm{Sch}_k^{\circ}: \mathrm{Aff}_k \to \mathfrak{Lrs}_k$

where Aff_k is the <u>category of affine algebraic spaces over k</u> , $\mathfrak{Lrs}_k$ is the category of locally ringed spaces (with $\mathcal{O}_X$ a structure sheaf of k-algebras), etc.

The absolute case (when k is not mentioned explicitly) must be reconsidered when we take $k = \mathbb{Z}$ (the initial object of $\mathcal{C}$, cf. Ch. I, §2).

In classical algebraic geometry k was a field (often algebraically closed, $k = \mathbb{C}$ during many pioneering years,...). Then $(\mathrm{Spec}\ k, \tilde{k})$ is reduced to a single point (0) and the field k "sitting" on (0) $((0),k)$. Moreover in classical times A used to be a finitely generated k-algebra: $A = k[\xi_1, \xi_2, \ldots, \xi_n]$.

12. EXAMPLES OF AFFINE SCHEMES. RECONSIDERATION OF NILPOTENT ELEMENTS.

We shall reconsider the cases of §5, pages 108,... . We already see how, by adding $\tilde{A}$ to Spec A we were able to recover A from the ℓ.r.s. $(\mathrm{Spec}\ A, \tilde{A})$. We saw this already in the trivial case $(\mathrm{Spec}\ k, k)$, where the underlying topological space is a one point set.

In the examples coming from an irreducible complex algebraic variety V if A does not have nilpotent elements the stalks at the "closed points" $\mathfrak{m} \in \mathrm{Max}\ A$ are the local rings $A_{\mathfrak{m}}$ of "rational functions" f/g $(f,g) \in A$ with $g(\mathfrak{m}) \neq 0$ $\Leftrightarrow g \neq 0 \bmod \mathfrak{m}$. The stalk at a point p which is not closed represents the local ring of rational functions which are not "∞" at the irreducible subvariety W_p represented by p ($g(x)$ is not identically zero at W).

When V is not irreducible, there is no "field of rational functions". This once again shows the advantage of the sheaf theoretic point of view which replaces the old birational point of view (useful only for irreducible varieties).

The stalk of $\tilde{\mathbb{Z}}$ in $(0) \in \mathrm{Spec}\ \mathbb{Z}$ is the field $\mathbb{Q}$ of the rationals $\mathfrak{m}_0 = 0$, thus the residue field at (0) is $\simeq \mathbb{Q}$. For any prime p, $\mathcal{O}_{X,p}$ is the ring of rational numbers a/b $(a,b \in \mathbb{Z})$ such that $b \neq 0 \pmod{p}$ $\mathfrak{m}_p$

is the ideal of $\mathcal{O}_{X,(p)}$, and $a \equiv 0 \mod p$ and the residue field $\kappa(p)$ is isomorphic with the prime field of characteristic p, $\mathbb{Z}_p$.

There is nothing especially remarkable about introducing a structure sheaf for a local ring $\mathcal{O}$.

The existence or nonexistence of nilpotent elements in A does not play any rôle in the construction of the structure sheaf $\tilde{A}$ and the fact that A is isomorphic with $\Gamma(X,\tilde{A})$ $(X = \operatorname{Spec} A)$ shows that the localizations of any $a \in A$ for all the $p \in \operatorname{Spec} A$ cannot always be equal to zero, unless $a = 0$. This shows that the interpretation of $\tilde{A}$ as a sheaf of germs of local functions, which was so useful at the beginning, is no longer legitimate in the general case: first of all the range of values $a(\mathfrak{m}_p) \in \kappa(p)$ changes from point to point (it is not a fixed field k as in the classical case of $\operatorname{Max} A$ (A a finitely generated k-algebra without nilpotent elements, k an algebraically closed field)). An incorrect consideration of the elements of the total space of $\tilde{A}$ as germs of functions applied to a non trivial nilpotent element $a \in A$ leads to the "paradox" of "functions which are not zero but whose value at any $x \in \operatorname{Spec} A$ is zero" $(\Leftrightarrow a \in p_x, \forall x \in \operatorname{Spec} A,$ $\Leftrightarrow a \in \operatorname{Nil} A)$.

13. QUASI-COHERENT SHEAVES. THE FUNCTOR $M \mapsto \tilde{M}$. We study the covariant functor $\sim: \mathfrak{M}_A \to \operatorname{Mod}(\tilde{A})$ between the category of A-modules $\mathfrak{M}_A$ and the category of $\mathcal{O}_X$-modules in $X = \operatorname{Spec} A$ $(X = \tilde{A})$. This is a very important instance of the "dictionary" between commutative algebra and ringed spaces.

GROTHENDIECK SUMMARY: We are going to make explicit the functorial correspondence $M \to \tilde{M}$ between A-modules M $\in \operatorname{Ob} \mathfrak{M}_A$ and sheaves of $\tilde{A}$-modules, $\tilde{M}$, on the Spectrum X of the ring A: $X = \operatorname{Spec} A = \{p \mid p \text{ prime ideal in } A\}$ viewed as a topological space(1) endowed with the sheaf of rings $\mathcal{O}_X = \tilde{A}$ (cf. §). We shall see that the category $\mathfrak{M}_A$ of A-modules can be identified with a full-subcategory of the category of all $\tilde{A}_x$-modules $(x \in X)$ over X, which is precisely the category of the so-called quasi-coherent sheaves over X.(2) This quasi-coherence is a purely local condition on the $\mathcal{O}_X$-modules of any ringed space $(X,\mathcal{O}_X)$ (cf. § 7). Namely, they are those which can be written locally as cokernels of homomorphisms of locally free modules over $\tilde{A}$.

This dictionary between A-modules and sheaves of modules on X is compatible with most things that we can think of. Namely, this functor commutes:

1) With finite sums, i.e. with finite products (<u>the functor $M \mapsto \tilde{M}$ is additive</u>)

2) With tensor products:

(13.1)
$$\widetilde{M \otimes_A N} \cong \tilde{M} \otimes_{\tilde{A}} \tilde{N}$$

3) With Hom:

(13.2)
$$\widetilde{\mathrm{Hom}_A(M,N)} \cong \mathrm{Hom}_{\tilde{A}}(\tilde{M},\tilde{N})$$

(<u>provided the first term is of finite presentation</u>). In addition:

4) It is exact, i.e.

(13.3)
$$0 \to M' \to M \to M'' \to 0 \text{ exact } \Rightarrow 0 \to \tilde{M}' \to \tilde{M} \to \tilde{M}'' \to 0 \text{ exact, and}$$

5) <u>It is fully faithful.</u>

The last statement means that <u>the category $\mathfrak{m}_A$ can be viewed as being a full subcategory of the category $\mathrm{Mod}(\tilde{A})$ of A-modules.</u>

One point is convenient to emphasize: <u>This functor $\sim$ does not commute with arbitrary products</u> $\prod_{i \in I}$ (I not necessarily finite). Precisely: it is not true that $\widetilde{\prod_{i \in I} M_i}$ is canonically isomorphic with $\prod_{i \in I} \tilde{M}_i$:

(13.4)
$$\widetilde{\prod_{i \in I} M_i} \ncong \prod_{i \in I} \tilde{M}_i$$

This comes from the fact that the property denied in (12.4) would be true iff for every $f \in A$ the homomorphism $(\prod_{i \in I} M_i)_f \to \prod_{i \in I} {M_i}_f$ would be an isomorphism for every f. This of course is not true! The localization is just a particular case of ring extension from A to A_f and a ring extension does not commute with products. Take for instance all the M_i to be equal to A. Then the question is whether $(A^I)f$ is naturally isomorphic with $(A_f)^I$. Let us check what this means. We have a family of elements of type $(\alpha_i/f^{n_i})_{i \in I}$ on one hand and on the other $(x/f^n)^I$ with a fixed exponent. It is clear that if I is infinite the exponents don't need to be bounded!

We should also be careful that formation of $\mathrm{Hom}(\ ,\ )$ does not commute with taking the tildas, i.e. in general we have an arrow

$$\mathrm{Hom}_A(M,N) \to \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\tilde{M},\tilde{N})$$

which in general is not an isomorphism. We can see this easily by taking $M = A^I$ with I infinite, just by reduction to the previous case. The "right" assumption, as we said in 2) (formula (12.2)) is that $\tilde{M}$ should be of finite presentation.

14. APPENDIX ON SHEAVES OF SETS.(1)

The following notes on sheaves of sets were delivered by GROTHENDIECK at the beginning lectures of his course on topoi. To include this in §7 would be too digressive; thus I prefer to include it in the Appendix, which should be particularly useful for readers with a prior knowledge of FAC; at the same time it would be helpful as an introduction to the abstract approach of GODEMENT's Bible.

I am going to talk about the <u>theory of topoi</u>. I like to see it as a kind of <u>generalization of classical general topology</u>. As a background we shall assume some familiarity with topological spaces, continuous maps, homeomorphisms, etc. etc. and on the other hand familiarity with the language of categories. Later we shall give some motivation for introducing something more general than topological spaces and give examples. But to understand the theory of topoi we shall also require some familiarity with the language of sheaves on a topological space. Now, I guess that this notion is not that familiar to everybody, so I will not assume anything known about it. I will review the standard theory of sheaves of sets(2) over topological spaces.

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(1) These notes were written with the collaboration of J. Winthrop.

(2) [GROTHENDIECK will consider mainly <u>sheaves of sets</u>, thus we shall omit this remark in the future. However later he will introduce various algebraic structures. The reader, knowing FAC, can take advantage of these lecture notes to strengthen his knowledge of sheaf theory by separating the topological properties from the algebraic ones.]

14.-1. PRESHEAVES OF SETS. Let X be a topological space. We consider the set $\mathfrak{G} = Op(X)$ of open subsets on X, i.e. the subset $Op(X)$ of the power set $\mathfrak{P}(X)$ (1) defining the topology on X. We recall that the axioms of a topology require that $Op(X)$ contain $\emptyset$ and X itself and be stable under arbitrary unions and finite intersections. $Op(X)$ is a partially ordered set (with the ordering defined by inclusion) and therefore $Op(X)$ already forms a category, by abuse of language. We denote this category by $\mathfrak{G}$ or $Op(X)$. As in any partially ordered set C if U,V are objects $U,V \in C$ the set of "homomorphisms" $Hom(U,V)$ from U to V is either empty if U is not contained in V or contains just the "inclusion map": $U \hookrightarrow V$ of U into V:

$$Hom(U,V) = \begin{cases} \emptyset & U \not\subset V \\ \to : U \to V & \end{cases} \tag{1.1}$$

The composition of arrows $U \to V \to W$ is defined in the obvious way. (We have no choice.) This particular construction of a category makes sense for any partially ordered C whatever; it does not use the fact that $C = Op(X)$.

In other words, the category $Op(X)$ has as objects the open sets of X and as arrows the graphs of the inclusion relations.(2)

A <u>presheaf</u> F on X is, by definition, a contravariant functor from the category $Op(X)$ to the category of sets. In other words F goes from the opposite category $\mathfrak{G}^*$ of $Op(X)$ to the category $Sets$ of sets.

Let us recall what that means:

1) To every object of the category, i.e. to every open set U of X, we associate a set $F(U)$, whose elements are called sections of F over

(1) [From the French part = subset: $\mathfrak{P}(X) = 2^X$.]

(2) This is true for the category attached to an ordered set $(S,<)$: Graph $< = \{(x,y) \in S \times S \mid x < y\}$. In our case $S = Op(X)$ and $<$ is the inclusion $\subset$.

U . (1)

2) to every inclusion $U \overset{i}{\hookrightarrow} V$ we associate a map:

(1.2) $$\rho_V^u : F(V) \to F(U)$$

between the corresponding sets (going in the opposite direction) where $\rho_V^u = F(i)$ is also denoted by the restriction symbol :

(1.3) $$F(V) \to F(V)\,|u = \rho_V^u(F(V))$$

and the following "evident axioms are satisfied

1) Transitivity: If another open set W contains V, i.e. $U \to V \to W$ are inclusions of open sets in X, then we have arrows $F(W) \to F(V) \to F(U)$ in the category of sets, preserving compositions. In other to the commutative diagram on the left (see below) corresponds a commutative diagram on the right

(1.4)
$$\begin{tikzcd} U \arrow[r, hook] \arrow[dr] & V \arrow[d] \\ & W \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} F(U) & F(V) \arrow[l] \\ & F(W) \arrow[u] \arrow[ul] \end{tikzcd}$$

In words:

Identity: F should transform identities into identities, i.e. to the identity map $U \overset{i_d}{\to} U$ corresponds the identity map $F(U) \to F(U)$ from $F(U)$ to itself.

The category $Presh(X) = Hom(\mathfrak{S}^\circ, Sets)$ of presheaves on X is defined as the category of all functors $\mathfrak{S}^\circ \to Sets$, ie. an object of $Presh(X)$ is a functor $F: \mathfrak{S}^\circ \to Sets$.

A homomorphism $F \overset{f}{\to} G$ from a presheaf F to a presheaf G (both over X) is, by definition of homomorphism of functors,(2) i.e. a collection

(1) This terminology comes from an old direct definition of sheaves over X, in terms of an étale covering space $S \overset{p}{\to} X$ (cf. next §). If U is open in X a section over U is a map $S: U \to S$ such that $ps = 1_u$.

(2) GROTHENDIECK prefer "homomorphism of functors" rather the synonimous "natural transformation", very common in the categorical jargon.

of maps $F(U) \to G(U)$ $(\forall U \in Ob(\mathcal{O}(X)))$ compatible with the restriction maps; i.e., for every open set U of X we have a map $F(U) \to G(U)$, such that the following diagram commutes

(1.6)

$$\begin{tikzcd} F(U) \arrow[r, "f(U)"] & G(U) \\ F(V) \arrow[u] \arrow[r, "f(V)"] & G(V) \arrow[u] \end{tikzcd}$$

where the vertical arrows are the restriction maps corresponding to U and V. Moreover, the composition $F \xrightarrow{f} G \xrightarrow{g} H$ of morphisms of presheaves is defined by considering in an obvious way the diagram

(1.7)

$$\begin{tikzcd} F(U) \arrow[r, "f(U)"] & G(U) \arrow[r, "g(U)"] & H(U) \\ F(V) \arrow[u] \arrow[r, "f(V)"] & G(V) \arrow[u] \arrow[r, "g(V)"] & H(V) \arrow[u] \end{tikzcd}$$

with all squares commutative.

This is the little "general nonsense" needed to construct the category Presh(X). So far we have not used the properties of the category $\mathcal{O}(X)$ of open sets we used just the properties of morphisms of functors = "natural transformations", but we will use them now to define a particular type of presheaves, the sheaves over X.

14.-2. SHEAVES OF SETS. Thus, we need to introduce some axioms on presheaves characteristic of sheaves.

We shall express these *axioms* in terms of two properties:

Let F be a presheaf on X for every open set U of X and for every open covering $\{U_i\}_{i \in I}$ of U $(\Leftrightarrow U = \bigcup_{i \in I} U_i)$ (1) we consider the restriction map of $F(U)$ into each of the $F(U_i)$ $(\forall i \in I)$ and therefore a map from $F(U)$ to the product of the $F(U_i)$

(2.1)

$$F(U) \to \prod_{i \in I} F(U_i)$$

(1) The union of the U_i can be defined in terms of the partial ordering in $Op(X)$ by the condition that U is the Supremum of the U_i for $i \in I$.

Then F is <u>separated</u> iff for any choice of U and of the cover: $\{U_i\}_{i\in I}$ the previous map $F(U) \to \prod_{i\in I} F(U_i)$ is <u>injective</u>.

Let us state this property in another way. First of all the set associated with any open $U \subseteq X$ be a presheaf F is called <u>the set c sections of F over U</u>[1], and for any inclusion $U \hookrightarrow V$ the restrictic $F(V) \to F(U)$ defines <u>the set of restricted sections</u>. Then the fact th presheaf F is separated means that for any open covering $\{U_i\}_{i\in I}$ o $U (\in Ob(Op(X)))$ <u>a section of F over U is known iff all of its restrictions to the U_i' are known</u> i.e. the arrow of (2.1) is an <u>injec arrow</u>, i.e. we can write instead of (2.1)

(2.2) $$F(U) \hookrightarrow \prod_{i\in I} F(U_i)$$

which means, in words, that <u>any section of F over U can be identifie with the collection of sections</u> of its restrictions $|U_i$ for every $i \in$

The second question arising, in characterizing sheaves as particula cases of presheaves, is whether any system of sections $\varphi_i = F(U_i)$, U_i open for every $i \in I$, can be obtained by restrictions from a section $F(U)$ over $U = \bigcup_{i\in I} U_i$. A necessary condition for such an $F(U)$ to ex is the "matching property":

(2.3) $$\varphi_i|U_i \cap U_j = \varphi_j|U_i \cap U_j$$

for every pair $(i,j) \in I \times I$. This is clearly necessary because of the transitivity property of the restriction maps.

[1] This terminology comes from an old direct definition of sheaves over X , in terms of an étale covering space $S \xrightarrow{p} X$ (cf. next §). If U is open in X a section over U is a map $S: U \to S$ such that $ps = 1_u$.

DEF. 1.1. We say that a presheaf F over X is a <u>sheaf</u> if for every $U \in \mathrm{Ob}\,\mathcal{O}(X)$ and every open covering of U the map (2.1) (which in general is not injective) is indeed injective (i.e. F is separated) and its image consists of all elements of $\prod F(U_j)$ satisfying the "matching property" (2.3) for every pair $(i,j) \in I \times I$.

We can write DEF. 1.1 in diagrammatic terms, as the condition that <u>the following sequence</u>

(2.4) $$F(U) \hookrightarrow \prod_{i \in I} F(U_i) \rightrightarrows \prod_{(i,j) \in I \times I} F(U_i \cap U_j)$$

<u>is exact</u>.

To interpret (2.4) we need to apply F to the two inclusions $U_i \cap U_j \to U_i$ and $U_i \cap U_j \to U_j$; thus we have the two arrows $F(U_i) \to F(U_i)|U_i \cap U_j$ and $F(U_j) \to F(U_j)|U_i \cap U_j$. The kernel of the double arrow $F(U_i) \rightrightarrows \prod_{(i,j) \in I \times I} F(U_i \cap U_j)$ means the system of $\prod_{i \in I} F(U_i)$ such that both restriction maps agree for every pair (i,j).

The usual meaning of exactness ($\Leftrightarrow$ Im = ker) is then verified in (2.3).

For two sheaves F and G a <u>sheaf morphism</u> $F \to G$ is by definition the same as a presheaf morphism between F and G. Thus, we can construct a category denoted by $\mathrm{Top}(X)$ (the category of sheaves over X) which is a <u>full subcategory</u> of the category $\mathrm{Presh}(X)$ of presheaves over X:

(2.5) $$\mathrm{Top}(X) \hookrightarrow \mathrm{Presh}(X) = \mathrm{Hom}(\mathcal{O}^*, \mathrm{Sets}).$$

Now it is time to give examples to show that <u>this notion of sheaf is a very natural one</u>; i.e. that <u>sheaves occur very frequently</u>.

EXAMPLES. Let E be any set and let us define a presheaf F on X by

(2.6) $$F(U) = \mathrm{Map}(U,E) = E^U \qquad \forall U \in \mathrm{Ob}(\mathrm{Op}(X))$$

i.e. $F(U)$ is the set of all maps from U to E.

If we have $U \hookrightarrow V$, we consider $F(V) = \mathrm{Map}(V,E)$ and the restriction map $F(V) \to F(U)$ is defined by restricting maps from V to E to maps from U to E, i.e. if $\varphi: V \to E$ belongs to $\mathrm{Map}(V,E)$ then $\varphi|U: U \to E$ belongs to $\mathrm{Map}(U,E)$. Since the restriction of maps is a transitive operation, we have certainly defined a presheaf. This presheaf F defined by (2.6) is in fact a sheaf. This means that whenever an open set U of X is covered by a family U_i $(i \in I)$ then to give a map from U to E amounts to the same as to give maps from U_i to E in such a way that these maps "match up" in the $U_i \cap U_j$ for every choice of $(i,j) \in I \times I$. In fact this would be even true if the U_i would not be open in X. Therefore we have a sheaf called the <u>sheaf of maps from X to E</u>.

Now, many sheaves which occur naturally in Mathematics are subsheaves of this one, but to explain that I need to define subpresheaves and subsheaves.

Let F,G be presheaves over X. Then we say that G is a subpresheaf of F, and we write $G \hookrightarrow F$ iff the two following conditions are true:

a) For every open $U \subset X$, $G(U) \hookrightarrow F(U)$ (i.e. $G(U)$ is a subset of $F(U)$)

b) For every inclusion $U \to V$ of open sets U,V the restriction map $G(V) \to G(U)$ is induced by $F(V) \to F(U)$, i.e. we have an obvious commutative diagram. This is what is called in general a subfunctor of a given functor.

In other words G is a subpresheaf of F iff it can be defined by a family of subsets $G(U)$ of the given $F(U)$, and the only condition that we need to impose is that the family be stable under the restriction maps.

Now let us assume that both F and G are sheaves over X . G is a subsheaf of F iff G is a subpresheaf of F . So a subsheaf of F consists of subsets $G(U)$ of the $F(U)$ which are stable under the restriction maps, according to the presheaf law, but now we need to make sure that the presheaf G is also a sheaf, thus we need an extra condition. It is evident that if G is a subpresheaf of a sheaf F , then G is separated, because if an open set U of X is covered by open U_i $(i \in I)$ as before, the $G(U_i)$ determine the $F(U_i)$, and $\prod_{i \in I} F(U_i)$ determines $F(U)$ because F is a sheaf, and $F(U)$ determines $G(U)$.

We have just proved that a subpresheaf of a separated presheaf is also separated. Now, what would it mean that G is not only separated but also G is a sheaf? This would mean that the map $U \to G(U)$ is of local type.[1] Summarizing: A subsheaf G of a sheaf F over X is defined by the collection $G(U) \subset F(U)$ $(U \in \mathfrak{Ob}(X))$, compatible with restriction maps and in which the property of a set belonging to the $G(U)$ of local nature.

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[1] We say that a property P of open sets of X is local if and only if for every open U of X , P holds in X if P holds for any open U_i of any open covering $\{U_i\}_{i \in I}$ of U .

EXAMPLE. Let us assume that the previous set E is a topological space and consider for each U the set $\mathrm{Cont}(U,E)$ of continuous maps from U to E,[1] where the composition of $p \circ s$ is the inclusion map of U <u>into X</u>.

Then $\mathrm{Cont}(U,E) \hookrightarrow \mathrm{Map}(U,E)$ and this property is compatible with the restriction maps (the restriction of a continuous map is continuous). Besides we know that continuity is a property of local character: A map: $U \to E$ is continuous if and only if all the restrictions $f|U_i$ to a family of open U_i covering U are continuous! Thus we have defined a subsheaf of the sheaf of maps from X to E, viz.: the sheaf of continuous local maps $U \to E$.

Assume now, for instance that X is a differentiable variety (of any fixed differentiability class C^r $(r \geq 1)$, C^∞) and that E is also a differentiable variety (of the same C^r), then we can consider the set of local differentiable maps $\mathrm{Diff}(U,E)$ and obtain a subsheaf of the previous sheaf of continuous local maps...[2]. It is well known that the differentiability property is of a purely local nature, preserved by restrictions: If U is covered by U_i then a map $f\colon U \to E$ is C^r-differentiable iff the $f|U_i\colon U_i \to E$ is C^r-differentiable for every $i \in$ for any open covering $\{U_i\}_{i\in I}$ of U!

3) Now we can extend this ad libitum, by taking for instance instead of differentiable varieties, analytic varieties,...or algebraic varieties,..., so any kind of "variety" defines a kind of sheaves...

4) Now there is still another kind of example of subsheaves of a sheaf in terms of fibre spaces:

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(1) This case contains the previous one of $\mathrm{Map}(U,E)$ if we endow E with the discrete topology.

(2) Sheaves of germs of local (continuous C^h, differentiable $h \geq 1$, analytic), etc.

Let us assume now that E is a fibre space over X, i.e. E,X are topological spaces and we consider a continuous $p: E \to X$ of E onto X.[1]

The triple (E,X,p) is a general continuous fibre space where X is the base space, E is the total space and p the projection.

Then for every open subset U of X let us look to the set $\Gamma(U, E/X)$ of all continuous maps $s: U \to E$ such that $p \circ s = 1_U$. Such maps s are commonly called <u>sections</u> of E over U (or just local sections if we do no want to mention U). For any inclusion $U \hookrightarrow V$ we have an induced map $\Gamma(V, E/X) \to \Gamma(U,E/X)$, i.e. the restriction maps transform V-sections in U-sections. Now it is very clear that a map $s: U \to E$ is a continuous section of U over X if and only if the restriction maps $s|U_i \to E$ are continuous sections over U_i for any choice of an open covering of U. <u>This sheaf is called the sheaf of local sections of the fibre space</u> $E \xrightarrow{p} X$. It is because of this particular situation that the name section of F over U was introduced.

There are any other examples suggested by the audience?

Schanuel suggested that it is convenient to point out that the previous example can be obtained from this one (sheaf of local continuous sections of a fibre space) by just considering the product $E \times X \ldots$, Grothendieck agrees.

If E is endowed with the discrete topology and we introduce the product topology in $X \times E$ then the previous sheaf can be interpreted also as a sheaf of local sections of $X \times E \xrightarrow{p} X$ (where $\to$ denotes the projection on the first factor). Then a section $s: U \to X \times E$ be identified with a function $U \to E$ $(x \to (x, f(e)))$, and this property is compatible with restriction maps,...

[1] p is not assumed to be onto.

Are there any further examples?...

.....Let me point out some more examples:

Let X be a locally compact topological space, and if we associate with any open set U of X the set of RADON measures U, this propert; is compatible with restriction maps. Besides this property is local and get a sheaf (the sheaf of Radon measures on open sets of the ℓ.c.s. X).

If X is a differentiable variety we can consider the sheaf of distributions on X (where a distribution is a continuous linear functio over vector spaces of local functions on X...). Distributions can be localized and we get a sheaf again.

Another example:

For every open set U of X let us assign the subsets of U clos in U. Then we have the transitivity property of restriction maps and we obtain a presheaf. This presheaf is a sheaf because $C \subset U$ is closed in U iff $C \cap U_i$ is closed in U_i for every covering of U. (i.e. the property of being closed is a local property).

In general any properties of subsets of an open set of a local nature enables to define a sheaf on X, for instance if X is an analytic variety we can define a sheaf of local analytic subsets of X.

Generally speaking one can say that sheaves are the most systematic tool to obtain global information, starting from local information, i.e. sheaves enables to "integrate" local data to global properties.

Now let $\mathcal{E} = (E,X,p)$ ($\mathcal{E}$ for short) be a fibre space over X. We associa to E a sheaf $\overline{E}$, called the sheaf of continuous local sections of E.

By associating with every fibre space E over X the sheaf $\tilde{E}$ we obtain a functorial correspondence $E \to \tilde{E}$. In other words if we have a morphism $E \to F$ of fibre spaces over X (that means a continuous map between total spaces making commutative the triangle below):

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & F \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

this allows us to define a morphism $\tilde{E} \to \tilde{F}$. In fact, whenever we have a section $s: U \to E$ on an open set U of X , we obtain a section $f \circ s$ of F over the same U . This maps $s \to f \circ s$ are compatible with restriction maps so we are going to obtain a homomorphism of sheaves $\tilde{f}: \tilde{E} \to \tilde{F}$. The map $f \to \tilde{f}$ for a variable f is compatible with composition of maps between fibre spaces over X and thus <u>we obtain a functor</u>:

(1.3) $$\mathrm{Fib}(X) \to \mathrm{Top}(X)$$

<u>form the category of fibre spaces over</u> X (denoted by Fib(X)) <u>to the category</u> Top(X) of sheaves over X .

The first question that might come to our minds is, can we reconstruct an object in Fib(X) just by knowing its image in Top(X)?, i.e. can we reconstruct a fibre space $\mathcal{E}$ over X in terms of $\tilde{E}$; (the corresponding sheaf). In formal terms we <u>would like to know</u> whether or not the functor (1.3) <u>is fully faithful</u>?

We shall see that this is not so, as we can convince ourselves quite readily. In order to see it let us consider the particular case when X is reduced to a single point e, $X = \{e\}$. Then the category of fibre spaces over a one-point space is just equivalent to the category of topological spaces, because for any such space E there is just one map $E \to \{e\}$.

Therefore $\mathrm{Fib}(e) \simeq \mathfrak{T}$ (category of all topological spaces). On the other hand, what is the category of sheaves $\mathrm{Top}\,[e]$ over a one-point space $[e]$? We consider again the maps associating with $[e]$ the secti $[e] \to E$. Up to a canonical equivalence a sheaf over a one-point space i known iff we know the corresponding E . Thus the functor (1.3) reduces, when $X = [e]$ to a functor: $\mathfrak{T} \to \mathrm{Sets}$ from the category of all topologi spaces to the category of sets, which associates with the topological spac E the underlying abstract set, i.e. we obtain a forgetful functor, in whi we just "forget" the topology of E! Now it is obvious that this functor is not fully faithful, i.e. we can't recover the topology of E from its underlying set. Therefore to give a fibre space $\mathcal{E}$ over X is something much more precise that to give the sheaf $\tilde{E}$ of local continuous sections!

14.-3.THE CATEGORY $\acute{E}t(X)$ OF ÉTALE COVERINGS OF X . We can wonder now whether or not we can define some full subcategory $(\acute{E}t(X))$ of the category $\mathrm{Fib}(X)$ of fibre spaces over X , such that the restriction to $\acute{E}t(X)$ of the functor (1.3), becomes fully faithful. For instance in the case of a one-point space $[e]$, which are the topological spaces whose topology is known if we know the corresponding underlying set? There are several choices. One of them would be to introduce the discrete topology: for a given set S there is just one discrete topology over S . Therefore if we take the restriction of the functor (1.3) to the category of discrete topological spaces over $[e]$ we obtain an equivalence of categories. Now we want to generalize this categorical equivalence to the general case of a general basis X . Thus we want to define a full subcategory of $\mathrm{Fib}(X)$

which in the $\{e\}$ case reduces to the category of discrete topological spaces over $\{e\}$. The property which looks "nice" is thus of a topological space E which is <u>étale over X</u>. We shall define it!

A continuous map $\pi: E \to X$ between topological spaces is called <u>étale</u> if it is a local isomorphism, in the following sense:

For every point $x \in E$ there exists an open neighborhood $V \subset E$ of x such that $\pi(V)$ is open in X such that π induces a map from V into $\pi(V)$ which is a homeomorphism, which means that π looks like a collection of local homeomorphism between some open sets of the space E upstairs and their projections $\pi(V)$ downstairs.

When this property holds we say also that π is an <u>étale morphism</u> between the topological spaces E,X. This is in fact a pretty old one in the theory of functions of one complex variable, where certain maps appear which are étale over an open subset of the complex plane.

Let us give some examples of "<u>étaleness</u>".

The most evident case is the inclusion map $U \to X$ of an open set U into X. This is the standard model!

Another example: Let us take a discrete topological space I, i.e. an abstract set endowed I endowed with its discrete topology and let us consider the product space $E = I \times X$. This means that we take the disjoint some of the copies $a \times X$ $(a \in I)$, which are open in E. Then the projection map $I \times X \to X$ reduces to the "identity map" $(a,x) \mapsto x$ on these open copies of X.

Now we shall construct the category $\text{Ét}(X)$ of étale covering spaces of X, which is a subcategory of $\text{Fib}(X)$. Let us look at the restriction functor (1.3) (going from Fib(X) to the category Top(X) of sheaves over X) to the subcategory $\acute{E}t(X)$. Then we get the generatication of th case of discrete spaces over a one point space $\{e\}$ that we were looking for! We obtain an <u>equivalence of categories</u>:

(3.1)
$$Et(X) \overset{\cong}{\to} Top(X)$$

on their own (étale coverings of $X \Leftrightarrow$ objects of $\acute{E}t(X)$] and we can jiggle back and forth between both languages. It turns out that for certai operations that we can perform on sheaves, the language of sections is by far the most convenient and in others the language of étale spaces is bette

EXAMPLES. Now maybe I should give some examples. Is there a sugges

QUESTION: Is there no translation in English for the French adjective *étale*?

Answer: No, this is a question that was raised about fifteen years ago. In French we say: <u>un espace étale dans un autre</u>..., which means a space "spread out over another", but in terms of a morphism, to say that it is "spread out" doesn't look good, so why not introduce another word into English...? Duskin asks why not say just a local homeomorphism? Grothendieck's answer is that when we deal in analogous contexts with differentiable, analytic or algebraic spaces we would like to use the same word, since the formal properties are the same (instead of introducing local diffeomorphisms, local biholomorphic maps, etc.). It is better to have a word which applies to all these particular cases...(1).

QUESTION: Is it true that the oldest examples of étale maps come from the construction of Riemann surfaces with several copies of the $\mathbb{C}$-plane?

Answer: Yes, provided you drop the branch points! All right, I will give this example: Let us take the map $f: \mathbb{C} \to \mathbb{C}$ of the affine complex line, in itself given by $x \to x^n$ $(n \geq 2)$. Then $f(0) = 0$. Then <u>the restriction</u> $f|C^*$ <u>onto itself</u> $(f(0) = 0)$, $\mathbb{C}^* = \mathbb{C} - \{0\}$ is <u>étale</u>. In fact the fibre over an $x \neq 0$ on the second plane is the set $x_1, x_2, \ldots, x_n$

(1) Later on in private conversation GROTHENDIECK told me that also in his native German the word <u>étale</u> is used,..instead of looking for a translation.

of the n n^{th}-roots of x in such a way that in choosing one of them, the others are obtained by multiplication with the n^{th} roots of unity $\exp(2\pi ikn^{-1})$ $(h = 0,1,\ldots,n-1)$. Then for any open neighborhood U of x not containing 0 obtain n copies of U covering U homeomorphically. Thus we see a difference in behavior of the map according to whether or not $x = 0$. The restriction to a neighborhood of zero is not étale (0 is a ramification point).

2) The previous example can be extended to any dominant morphism $X \xrightarrow{f} Y$ ($\Leftrightarrow$ $f(X)$ is dense in Y) between two complex irreducible non singular curves. Throwing out finitely many points of Y (ramification points) the restriction of the map f to $X - \bigcup_R f^{-1}(R)$ (R-ramification points) is étale.

3) A third example of covering étale everywhere is the covering map $\tilde{X} \to X$ of a topological connected manifold by its universal covering space.

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